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SOME EXAMPLES OF BOREL SETS

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The aim of this note is to give a simple proof that for every countable ordinal a there exist Borel subsets of the Hilbert cube H which are exactly of the class a, i.e. are of the class a but not of a class $\beta < a$.

More precisely the third of us has defined [2] in a simple way some subsets M_a, A_a of H which are exactly of the multiplicative and additive class a, respectively. However, in [2] the existence of Borel sets of arbitrarily high classes (in a metric space) was assumed to be known. The first of us observed that the required property of M_a and A_a can be deduced from the Brouwer fixed-point theorem for the Hilbert cube H, and the second of us simplified this proof. Consequently, we are able to give a very simple construction of Borel sets of arbitrarily high classes together with a very simple direct proof of this property (see theorem (iii) below). For the sake of completeness a part of [2] is reproduced (theorems (i) and (ii)).

The final part, due to the first of us, contains a new construction of universal functions for Borel sets of an additive or multiplicative class a (see [1], vol. I, p. 278). This construction gives also a proof that there exist Borel sets which are exactly of the class a. Combining this fact with the property (ii) of M_a and A_a we get another direct proof that M_a and A_a are exactly of the class a. This proof makes no use of the Brouwer theorem.

First, we recall the inductive definition of the sets M_a and A_a.

Let M_0 be a one-point subset of H and $A_0 = H \setminus M_0$. Suppose the sets M_ξ and A_ξ to be defined for all ordinals $\xi < a$. If $a = \beta+1$ let

$$M_a = A_\beta \times A_\beta \times A_\beta \times \ldots \subset H^{\aleph_0},$$

if a is a limit ordinal, let

$$M_a = A_1 \times A_2 \times \ldots \times A_\xi \times \ldots = \mathbf{P}_{\xi<a} A_\xi \subset H^{\aleph_0}.$$

Since $H^{\aleph_0}$ is homeomorphic with H, we may consider M_a as a subset of H. Let $A_a = H \setminus M_a$.

It follows immediately from the definition that

(i) *M_a is a Borel set of the multiplicative class a in H; A_a is a Borel set of the additive class a in H.*

The following theorem, proved in [2], expresses a remarkable universal property of the sets M_a and A_a:

(ii) *If X is a metric space and $B \subset X$ is a Borel set of the multiplicative (additive) class a in X, then there exists a continuous mapping $\varphi\colon X \to H$ such that $\varphi^{-1}(M_a) = B$ (such that $\varphi^{-1}(A_a) = B$).*

The proof is by induction. In the case $a = 0$ this is obvious. Suppose that (ii) is true for $a < a_0$, $a_0 = \beta+1$, and $B \subset X$ is a Borel set of the multiplicative class a_0, i.e. $B = B_1 \cap B_2 \cap B_3 \cap \ldots$, where $B_n \subset X$ is a Borel set of the additive class β for $n = 1, 2, 3, \ldots$ By the induction hypothesis, there exists a continuous mapping $\varphi_n\colon X \to H$ such that $\varphi_n^{-1}(A_\beta) = B_n$. The continuous mapping $\varphi\colon X \to H^{\aleph_0} = H$, defined by the formula $\varphi(x) = \{\varphi_n(x)\}$, has the property $\varphi^{-1}(M_{a_0}) = B$. If a_0 is a limit ordinal, the proof is similar.

By passage to complements, we prove the lemma for Borel sets B of the additive class a.

It is easily seen that (ii) is also true for a perfectly normal X.

(iii) *M_a is not of the additive class a in H; A_a is not of the multiplicative class a in H.*

It is enough to prove the first part of (iii).

Suppose that M_a is of the additive class a. By (ii) there exists a continuous mapping $\varphi\colon H \to H$ such that $\varphi^{-1}(A_a) = M_a$. The continuous mapping φ transforms M_a into a subset of A_a and A_a into $H \setminus A_a = M_a$. Hence the equality $\varphi(x) = x$ does not hold for any point $x \in H$. This contradicts the Brouwer fixed-point theorem for H (see [1], vol. II, p. 263).

We recall (see [1], vol. I, p. 275) that by a universal function of the multiplicative (additive) class a for a metric space X we understand any mapping Φ from a metric space Y into the class of all subsets of X, such that

a) *Φ maps Y onto the class of all Borel subsets of X of the multiplicative (additive) class a,*

b) *the set $\{(x, y) \in X \times Y\colon x \in \Phi(y)\}$ is a Borel subset of $X \times Y$ of the multiplicative (additive) class a.*

In the sequel we shall take for Y the space H^X of all continuous mappings of X into H metrized by the formula

$$\bar{\varrho}(f, g) = \sup_{x \in X} \varrho\big(f(x), g(x)\big),$$

where ϱ is a metric for H.

(iv) *The mapping*

$$\Phi(\varphi) = \varphi^{-1}(M_a), \quad \textit{where} \quad \varphi \in H^X,$$

is a universal function of the multiplicative class a for a metric space X.

The property a) follows directly from (i) and (ii). To prove b) let us observe that the mapping $v: X \times H^X \to X$ defined by the equation $v(\varphi, x) = \varphi(x)$ is continuous and the set

$$\{(x, \varphi) \in X \times H^X : x \in \Phi(\varphi)\}$$
$$= \{(x, \varphi) \in X \times H^X : x \in \varphi^{-1}(M_a)\} = v^{-1}(M_a)$$

is of the multiplicative class a.

In the case of separable metric spaces there exists a common parameter space for universal functions:

(iv') *For each subspace X of H the mapping*

$$\Phi(\varphi) = X \cap \varphi^{-1}(M_a), \quad \textit{where} \quad \varphi \in H^H,$$

is a universal function of the multiplicative class a for X.

The property a) follows from (i) and (ii), and the property b) from the formula

$$\{(x, \varphi) \in X \times H^H : x \in \Phi(\varphi)\}$$
$$= \{(x, \varphi) \in H \times H^H : x \in X \cap \varphi^{-1}(M_a)\} = (X \times H^H) \cap v^{-1}(M_a),$$

where $v: H \times H^H \to H$.

Obviously, setting A_a instead of M_a in (iv) and (iv') we obtain universal functions of the additive class a.

Since H^H is a metric separable space (see [1], vol. I, p. 120), there exists a homeomorphism i of H^H into H. Putting $X = i(H^H)$, we conclude from (iv') that the set

$$\{(g, f) \in H^H \times H^H : f(\bar{g}) \in M_a\},$$

where $\bar{g} = i(g)$, is of the multiplicative class a in $H^H \times H^H$. The intersection of this set with the diagonal of $H^H \times H^H$ is of the same class; the same holds, therefore, for the set

$$M_a^* = \{f \in H^H : f(\bar{f}) \in M_a\} \subset H^H.$$

(v) *The set M_a^*, of the multiplicative class a, is not of the additive class a.*

Indeed, if it were, then by (ii) there would exist a continuous mapping $\varphi: H \to H$ such that $i(M_a^*) = i(H^H) \cap \varphi^{-1}(A_a)$ and we would have for every $f \in H^H$:

$$\big(f(\bar{f}) \in M_a\big) \equiv \big(\varphi(\bar{f}) \in A_a\big).$$

But for $f = \varphi \epsilon H^H$ this gives

$$\big(\varphi(\bar{\varphi}) \epsilon M_a\big) = \big(\varphi(\bar{\varphi}) \epsilon A_a\big),$$

which is a contradiction.

By the same argument as in [2], it follows from (v) that M_a is not of the additive class a. In fact, there exists a continuous mapping $\varphi: H^H \to H$ such that $\varphi^{-1}(M_a) = M_a^*$. Thus, the hypothesis that M_a is of the additive class a would imply that so is M_a^*, in contradiction with (v). By passing to complements we infer that A_a is not of the multiplicative class a.

Observe that replacing the Hilbert cube H by the Cantor set C we obtain also sets M_a and A_a satisfying (i) and (iii). Indeed, (ii) remains true under the additional hypothesis that $\dim X = 0$, hence (iv′) is valid for subspaces of C. Since C^C is topologically contained in 2^C (see [1], vol. I, p. 111) and thus in C, theorems (v) and (iii) hold.

REFERENCES

[1] C. Kuratowski, *Topologie I, II*, Warszawa 1958 and 1961.

[2] R. Sikorski, *Some examples of Borel sets*, Colloquium Mathematicum 5 (1958), p. 170-171.

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