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A MODEL OF HYPERBOLIC STEREOMETRY BASED ON THE ALGEBRA OF QUATERNIONS

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Let R, C and H denote fields of reals, complex numbers and quaternions, respectively. We consider the multiplicative group on $C\times H$ which acts on $H_*^2 := H\times H\setminus\{(0,0)\}$ by the rule

$$(1)\qquad (C\times H)\times H_*^2 \to H_*^2,\qquad \big((z,h),(x^1,x^2)\big)\mapsto (zx^1h, zx^2h).$$

Denote by N the space of orbits of H_*^2 under this action. N is a basic space which we shall provide with a hyperbolic metric. It is known that the hyperbolic stereometry can be considered as the Riemannian geometry with the basic manifold $R_+^3 := \{(x^1,x^2,x^3)\mid x^3>0\}$ and with the fundamental metric form

$$ds^2|_{(x^1,x^2,x^3)} = \frac{K^2}{(x^3)^2}\big((dx^1)^2+(dx^2)^2+(dx^3)^2\big).$$

We are going to obtain this metric and some other properties of the hyperbolic space by considering N as a base of a certain Klein space. An analogical treatment has been performed in a 2-dimensional case and resulted in a brief and consequent system of analytical geometry of the Lobachevski plane ([2], cf. also [4]).

I. THE FUNDAMENTAL GROUP

1. Proposition. *The space N is a compact 3-dimensional manifold with a boundary.*

Proof. Orbits of the action

$$H\times H_*^2 \to H^2 \,/\, \big(a, (x^1,x^2)\big)\mapsto (x^1a, x^2a)$$

are points of the projective space over H, denoted by PH (cf. [1]). This space can be covered by two charts defined by mappings

$$\mu_1\colon \{x^1, x^2\} \mapsto x^1(x^2)^{-1} \quad \text{and} \quad \mu_2\colon \{x^1, x^2\} \mapsto x^2(x^1)^{-1},$$

where $\{x^1, x^2\}$ denotes an image of (x^1, x^2) under the canonical projections $H^2_* \to PH$. The real dimension of PH is 4. Thus N can be viewed as the space of orbits in PH under the action

$$C \times PH \to PH \,/\, (z, \{x^1, x^2\}) \mapsto \{zx^1, zx^2\}.$$

We have

$$\mu_i\{zx^1, zx^2\} = z(\mu_i\{x^1, x^2\})z^{-1}, \quad i = 1, 2.$$

We shall see what are orbits in H under the action $a \mapsto zaz^{-1}$, where $z \in C$ and $a \in H$. For that purpose write a in the form $a = a' + a''j$, where a' and a'' are complex numbers and j is the third unity in H. Note that this decomposition can be obtained as follows:

$$(2) \qquad a' = \frac{1}{2}\big(a + ia(-i)\big), \quad a'' = \frac{1}{2}\big(a - ia(-i)\big)(-j).$$

Thus

$$zaz^{-1} = za\bar{z}|z|^{-2} = a' + \exp(i2\arg z)a''j.$$

Since a' and $|a''|$ are invariant, we can define mappings m_1 and m_2 as follows: if $(x^1, x^2) \in H$ and $x^2 \neq 0$, then

$$(3) \qquad \mu_1\{x^1, x^2\} = x^1(x^2)^{-1} = h' + h''j, \quad \text{where } h', h'' \in C.$$

We put

$$m_1\big(p(x^1, x^2)\big) = (\mathrm{re}\,h', \mathrm{im}\,h', |h''|),$$

where p denotes the canonical projection of H^2_* onto N. Thus the values of m_1 lie in $\mathrm{cl}\,R^3_+$. Similarly, we define

$$m_2\colon \{p(x^1, x^2) \,|\, x^1 \neq 0\} \to \mathrm{cl}\,R^3_+,$$

where

$$p(x^1, x^2) \mapsto \Big(\mathrm{re}\big(\mu_2(x^1, x^2)'\big), \mathrm{im}\big(\mu_2(x^1, x^2)'\big), |\mu_2(x^1, x^2)''|\Big),$$

and $\mu_2(-) = \mu_2(-)' + \mu_2(-)''$ is a decomposition analogous to (3).

We see that N can be covered by two local charts, m_1 and m_2 being the corresponding mappings. The boundary of N is $\{p(x^1, x^2) \,|\, x^1(x^2)^{-1} \in C\} \cup \cup \{\infty\}$. This completes the proof.

2. Note that N is homeomorphic to a manifold the points of which are circles in the C-plane, including circles with radius equal to 0 and ∞. These singular circles constitute the boundary.

Denote by L the group of non-singular complex 2×2-matrices. We map L onto a transformation group T which acts on N as follows:

(4) if $a = [a^i_k]_{i,k=1,2} \in L$ and $u = p(x^1, x^2) \in N$, then

$$\tau_a u := p(a^1_1 x^1 + a^1_2 x^2,\ a^2_1 x^1 + a^2_2 x^2).$$

Observe that τ_u does not depend on the choice of the initial point (x^1, x^2) on the orbit u. Thus (4) defines the action correctly and we denote by T the image of L by τ.

We have $\tau_a = \tau_b$ if and only if $b = \lambda a$ for some $0 \neq \lambda \in C$. This implies the following

3. Proposition. *T is isomorphic to the group of complex 2×2-matrices with the determinant equal to 1. The real dimension of T is 6.*

4. Theorem. *Let c denote the point in N with the m_1-coordinates $(0, 0, 1)$. Then a stationary subgroup $S \subset T$ of c consists of the matrices of the form*

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

such that $a\bar{a} + b\bar{b} = 1$.

Proof. Consider the equation $(aj+b)(cj+d)^{-1} = j$ and split it into the $'$ and $''$ parts according to (3). We obtain $c = -\bar{b}$ and $d = \bar{a}$. Then we normalize the obtained matrices according to proposition 3.

Let us denote by Λ the closure of the set $\{u \in N \mid \mu_1(u) = (0, 0, t)\}$. Denote by T_Λ the stationary group of Λ.

5. Theorem. *T_Λ consists of the matrices of the form*

$$\begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & a \\ -1/a & 0 \end{bmatrix}, \quad \text{where } a \in R.$$

The proof is analogous to that of theorem 4.

6. Proposition. *The group S is isomorphic to the group of rotations of the Euclidean 3-dimensional space.*

This fact can be proved by checking that the Lie algebras of both groups are isomorphic.

7. Lemma. *Let $h = h' + h''j \in H$ be such that $h' \neq 0$ and $h'' \neq 1$. Then there exist two complex numbers g_1 and g_2 and two complex singular matrices G_1 and G_2 such that*

(i) $|g_1| \neq 1$, $|g_1 g_2| = 1$, *and* $\arg g_1 = \arg g_2 = \arg h'$;

(ii) *each G_ν sends h to g_ν and j to itself for $\nu = 1, 2$.*

Proof. In view of theorem 4, we have to find g_ν and the required matrices from the equation

$$a(h'+h''j)+b = gj\big(-\bar{b}(h'+h''j)+\bar{a}\big).$$

After performing some simple calculations and splitting both member: into their $'$ and $''$ parts, we obtain the following system of equationss

$$h'a+(1-g\bar{h}'')b = 0, \quad (h''-g)a+g\bar{h}'b = 0.$$

Its determinant must be 0 and for g we have the equation

$$-\bar{h}''g^2+(1+\bar{h}'h'+\bar{h}''h'')g-h'' = 0 \tag{5}$$

which can be written in the form

$$(\gamma+\gamma^{-1})/2 = (1+\bar{h}'h'+\bar{h}''h'')/(2|h''|), \tag{6}$$

where $\gamma = g\exp(-i\arg h'')$. Since γ is real and it satisfies the equation

$$\operatorname{ch}(\log\gamma) = (1+|h|^2)/(2|h''|), \tag{7}$$

there exist two distinct roots of equation (5), namely $g_1 = \gamma_1 e^{ia}$ and $g_2 = \gamma_2 e^{ia}$, where $a = \arg h''$. These roots yield the two matrices

$$\begin{bmatrix} g_\nu\bar{h}' & g_\nu - h'' \\ \bar{h}''-\bar{g}_\nu & \bar{g}_\nu h' \end{bmatrix} \quad (\nu = 1, 2)$$

which satisfy (ii).

8. THEOREM. *For any two points $u, v \in \operatorname{int} N$, there exist two elements a_1 and a_2 in T such that each a_ν sends u to $p(j, 1)$ and v to $p(g_\nu j, 1)$, where g_ν are complex and $|g_1 g_2| = 1$.*

Proof. Write $u = p(w'+w''j, 1)$. Then the matrix

$$a_0 = \begin{bmatrix} 1 & w' \\ 0 & w'' \end{bmatrix}$$

sends u to $p(j, 1)$. We choose an h such that $a_0 v = p(h, 1)$ and apply lemma 7. This implies the existence of G_1 and G_2 which send $a_0 u$ to itself and $a_0 v$ to $p(g_1 j, 1)$ (or, respectively, to $p(g_2 j, 1)$), where g_1 and g_2 satisfy the conditions of the theorem. Then we put $a_1 = G_1 \circ a_0$ and $a_2 = G_2 \circ a_0$.

9. PROPOSITION. *The stationary group of the pair $\big(p(j, 1), p(gj, 1)\big)$, where $1 \neq g \in C$, is represented by matrices of the form*

$$\begin{bmatrix} e^{\tau i} & 0 \\ 0 & e^{-\tau i} \end{bmatrix}.$$

The action of this group can be expressed also by

$$p(h, 1) \mapsto p(e^{\tau i} h e^{\tau i}, 1).$$

This proposition can be easily obtained by lemma 7. The same lemma allows us to prove the following

10. PROPOSITION. *The group T acts transitively on a bundle of directions on N.*

II. THE CROSS-RATIO AND THE METRIC IN N

We recall that Λ is the orbit of the point $p(j, 1)$ under the action of the group of matrices of the form

$$\begin{bmatrix} \sqrt{s} & 0 \\ 0 & 1/\sqrt{s} \end{bmatrix},$$

where s varies in the half-line of positive numbers (cf. theorem 5). We observe that these matrices constitute a connected component of T_Λ. However, T_Λ contains another topologically connected component to which matrices of the form

$$\begin{bmatrix} 0 & -\sqrt{s} \\ \sqrt{s} & 0 \end{bmatrix}$$

belong. Each such matrix can be represented as a product

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{s} & 0 \\ 0 & -1/\sqrt{s} \end{bmatrix}.$$

An action of the first matrix of this decomposition is nothing but a change of orientation on N. More precisely, it sends $p(j, 0)$ to $p(0, j)$ and *vice versa*.

Then we can define the cross-ratio on $\operatorname{int} N$ which is invariant under T_Λ. We denote by $(u, v; r, q)$ the value of the cross-ratio of the quadruple (u, v, r, q). If $u = p(a_1 j, 1)$, $v = p(a_2 j, 1)$, $r = p(\beta_1 j, 1)$ and $q = p(\beta_2 j, 1)$, then we have

$$(u, v; r, q) = \frac{a_1 - \beta_1}{a_2 - \beta_1} \Big/ \frac{a_1 - \beta_2}{a_2 - \beta_2}.$$

We extend this function onto $\operatorname{cl} \Lambda$ by continuity. In particular, we have

$$(8) \qquad \big(u, v; p(0, 1), p(1, 0)\big) = a_1/a_2.$$

Thus we obtain

11. PROPOSITION. *We have*

$$\big(u, v; p(0, 1), p(1, 0)\big)\big(v, w; p(0, 1), p(1, 0)\big) = \big(u, w; p(0, 1), p(1, 0)\big)$$

and, if v lies between u and w, then

$$\left|\log(u, v; p(0,1), p(1,0))\right| + \left|\log(v, w; p(0,1), p(1,0))\right| = \left|\log(u, w; p(0,1), p(1,0))\right|.$$

Now we are able to define a metric in N.

12. Definition. We say that three distinct points in N are *N-collinear* if there exists $t \in T$ which maps these three points to points of Λ. The *N-line* through points u and v is the set of points which are collinear with u and v.

13. PROPOSITION. *Every N-line is a curve in N. Its boundary consists of two points in the boundary of N.*

14. Definition. Fix a positive real K. We define a *T-invariant distance* δ in N as follows. If u and v are distinct points in $\operatorname{int} N$, then we choose $a \in T$ such that $au = p(j, 1)$ and $av = p(gj, 1)$, where $g \in C$, and we set

$$\delta(u, v) := K\left|\log|g|\right|.$$

In view of sections 8 and 9, the function δ is uniquely determined. We have to express it in m_1-coordinates. If $u = p(c, 1)$ and $v = p(h, 1)$, then we construct a transformation a according to the proof of theorem 8. Then we use formula (5) and theorem 8. After some calculations we obtain

$$\operatorname{ch}\left(K^{-1}\delta(u, v)\right) = \left(|f' - h'|^2 + |f''|^2 + |h''|^2\right) / |2f''h''|. \tag{9}$$

This formula implies immediately $\delta(u, v) = \delta(v, u)$. Additivity of δ on any N-line follows from proposition 9.

15. THEOREM. *The infinitesimal form of the metric δ is*

$$ds^2|_{p(h,1)} = \frac{K^2}{(x^3)^2}\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right), \tag{10}$$

where $x^1 = \operatorname{re} h'$, $x^2 = \operatorname{im} h'$, $x^3 = |h''|$.

Proof. Consider a curve which is parametrized by the mapping $t \mapsto p(h + tx, 1)$. Let X be a 1-jet of this mapping, its source being 0. Thus X is a vector which is tangent to N at $u = p(h, 1)$. We have to calculate the norm $|X|$ of X, which is induced by δ. We have

$$|X| = \lim_{t \downarrow 0} \frac{1}{t}\,\delta\left(p(h + tx, 1), p(h, 1)\right).$$

After some elementary calculations we obtain

$$|X| = \frac{K^2}{|h''|^2}\left((\mathrm{re}\,x')^2+(\mathrm{im}\,x')^2+|x''|^2\right)$$

which is consistent with (9). So the following theorem is a corollary to the just obtained formulas:

16. THEOREM. *δ is a hyperbolic distance.*

III. FINAL REMARKS

Let us denote by B the boundary of N. This boundary is homeomorphic to the complex projective line (which is isomorphic to the real Moebius sphere). Each N-line has exactly two points in common with B. These are the so-called infinite points of the line. Conversely, each pair of distinct points on B determines exactly one N-line.

The group T acts as a group of projective transformations. The proper and improper circles in B are traces of N-planes according to the following definition:

17. Definition. A subset $\Pi \subset N$ is called an *N-plane* if there exists $a \in T$ which maps Π to

$$\Pi_0 = \mathrm{cl}\{w \in N \mid w = p(h, 1), \text{ where } \mathrm{im}\,h' = 0\}.$$

Thus each N-plane is a 2-dimensional submanifold of N.

18. THEOREM. *For any N-lines a and β with the unique point of coincidence $v \in \mathrm{int}\,N$, there exists a unique N-plane Σ such that $a \subset \Sigma$ and $\beta \subset \Sigma$.*

Proof. By theorem 8, there exists $a \in T$ which sends a to Λ. Let z_1 and z_2 be the infinite points of the N-line $a\beta$. Apply proposition 9 and perform a transformation r such that $r\Lambda = \Lambda$ and $\mathrm{im}\,z_1 = \mathrm{im}\,z_2 = 0$. Hence $r \circ a$ sends a and β into Λ. Thus $a^{-1} \circ r^{-1}\Lambda$ is the N-plane through a and β.

The following two theorems are easy to prove.

19. THEOREM. *The stationary subgroup T_0 of Π_0 consists of those transformations which have real matrices and determinants equal to* 1.

20. THEOREM. *If we restrict the Klein space (N, T) to (Π_0, T_0), then we obtain the plane hyperbolic geometry.*

Let a and β be two N-lines as in theorem 18. We denote by z_1, z_2 and s_1, s_2, respectively, pairs of their infinite points. Thus z_1, z_2, s_1, s_2 are situated on a circle in B. We denote by ϑ hyperbolical measure of the angle between α and β. Observe that, in view of proposition 6, the mapping $m_1 | \mathrm{int} N$ is conformal. Then the following relation holds between ϑ and the cross-ratio of the chosen pairs of infinite points:

$$(z_1, z_2; s_1, s_2) = -\mathrm{ctg}\frac{\vartheta}{2}.$$

This is proved in [3] in the case where α and β are both in Π_0, but remains true in general because of the invariance with respect to T of both members of this equality.

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