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UNIFORM LIMITS OF DARBOUX FUNCTIONS ON THE EUCLIDEAN PLANE

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1. Introduction. Let X be a topological space with a base $\mathscr{B}$. Misik defined two classes, $D(\mathscr{B})$ and $D_0(\mathscr{B})$, of real-valued functions on X (see [4] and [5]). $f \in D(\mathscr{B})$ if for each $B \in \mathscr{B}$, x, $y \in \bar{B}$ (the closure of B), and each real number η such that $f(x) < \eta < f(y)$, there exists $z \in B$ with $f(z) = \eta$. $f \in D_0(\mathscr{B})$ if for each $B \in \mathscr{B}$, x, $y \in \bar{B}$, each real number η such that $f(x) < \eta < f(y)$, and each $\varepsilon > 0$, there exists $z \in B$ with $f(z) \in (\eta - \varepsilon, \eta + \varepsilon)$.

In the case where X is the Euclidean plane E_2 and $\mathscr{B}$ is the collection of all open intervals in E_2, the following is proved [6]:

THEOREM 0. *Let $f \in D_0(\mathscr{B})$. Then $\max(f, g) \in D_0(\mathscr{B})$ for every $g \in D_0(\mathscr{B})$ if and only if f is upper semi-continuous on E_2.*

It is natural to ask if the same can be said for the class $D(\mathscr{B})$. The answer turns out to be affirmative. To prove it, we need to use an intermediate class of functions which is exactly the class of uniform limits of functions in $D(\mathscr{B})$. This concept, while real-valued functions on the real line are considered, can be found in [2].

2. Preliminaries. Throughout this paper, all functions are real valued, c is the cardinality of continuum, X is a topological space, and if $A \subset X$, then $\bar{A}$, A', A° and $\operatorname{Card}(A)$ denote the closure, derived set, interior and the cardinality of A, respectively. Also, $\mathscr{B}$ is a base for X such that $\operatorname{Card}(B) \geqslant c$ for each $B \in \mathscr{B}$.

DEFINITION. Let $A \subset X$. The set A is said to be *dense* ($\mathscr{B}$) [*c-dense* ($\mathscr{B}$)] *in itself* if, for each $x \in A$ and $B \in \mathscr{B}$ with $x \in \bar{B}$,

$$B \cap A \neq \emptyset \quad [\operatorname{Card}(B \cap A) \geqslant c].$$

If $A_0 \subset A$, A_0 is said to be *dense* ($\mathscr{B}$) [*c-dense* ($\mathscr{B}$)] *in* A if, for each $x \in A$ and $B \in \mathscr{B}$ with $x \in \bar{B}$,

$$B \cap A_0 \neq \emptyset \quad [\operatorname{Card}(B \cap A_0) \geqslant c].$$

Let $A_0 \subset A \subset X$. It is clear that if A is dense ($\mathscr{B}$) [c-dense ($\mathscr{B}$)] in itself and A_0 is dense [c-dense] in A, then A_0 is dense ($\mathscr{B}$) [c-dense ($\mathscr{B}$)] in A and in itself.

The proof of the following proposition is omitted. Boboc and Marcus proved a similar result [1].

PROPOSITION. *Let $X = E_2$ and $\mathscr{B}$ be the collection of all open intervals in E_2. Then any c-dense ($\mathscr{B}$) in itself set A is the union of countably infinitely many disjoint nonempty subsets each of which is c-dense ($\mathscr{B}$) in A.*

3. Definition and some properties of $D_u(\mathscr{B})$.

DEFINITION. A function f on X is said to be in the *class* $D_u(\mathscr{B})$ if, for each $B \in \mathscr{B}$, $x, y \in \bar{B}$, each real number η such that $f(x) < \eta < f(y)$, and each $\varepsilon > 0$,

$$\operatorname{Card}(\{z \in B: f(z) \in (\eta-\varepsilon, \eta+\varepsilon)\}) \geqslant c.$$

This is equivalent to $f \in D_u(\mathscr{B})$ if, for each $B \in \mathscr{B}$, $x, y \in \bar{B}$, and any real numbers a, b such that $f(x) \leqslant a < b \leqslant f(y)$,

$$\operatorname{Card}(\{z \in B: f(z) \in (a, b)\}) \geqslant c.$$

Clearly, $D(\mathscr{B}) \subset D_u(\mathscr{B}) \subset D_0(\mathscr{B})$. In [3], Farková proved a theorem under the condition that $\mathscr{B}$ satisfies (1*) and (2).

(1*) For arbitrary $x \in X$, $B \in \mathscr{B}$, if O is an open set and $x \in O \cap \bar{B}$, then there exists $U \in \mathscr{B}$ such that

$$U \subset O \cap B \quad \text{and} \quad x \in \bar{U} - U.$$

(2) For every $B \in \mathscr{B}$ and every decomposition of B,

$$B = C \cup D, \quad C \cap D = \emptyset, \quad C \neq \emptyset \neq D,$$

with the property that $\bar{U} \cap B \subset C$, $\bar{U} \cap B \subset D$, respectively, whenever $U \in \mathscr{B}$ and $U \subset C$, $U \subset D$, respectively,

$$C' \cap D \neq \emptyset \neq C \cap D'.$$

THEOREM F ([3]). *Let $\mathscr{B}$ be a base for X satisfying* (1*) *and* (2). *Let $f, g \in D_0(\mathscr{B})$ be such that every $x \in X$ is a point of upper semi-continuity of f or g. Then*

$$\varphi = \max(f, g) \in D_0(\mathscr{B}).$$

It is not hard to modify the proof and obtain the following

THEOREM 1. *Let $\mathscr{B}$ be a base for X satisfying* (1*) *and* (2). *Let $f, g \in D_u(\mathscr{B})$ [or $D(\mathscr{B})$] be such that every $x \in X$ is a point of upper semi-continuity of f or g. Then $\varphi = \max(f, g) \in D_u(\mathscr{B})$ [or $D(\mathscr{B})$, respectively].*

THEOREM 2. *Let F be a continuous function of two real variables. If the property* (P_0) *below holds, then the property* (P_u), *which is obtained by replacing $D_0(\mathscr{B})$ by $D_u(\mathscr{B})$ in* (P_0), *also holds.*

(P_0) *If $f, g \in D_0(\mathscr{B})$ are such that each $x \in X$ is a point of continuity of f or g, then $F(f, g) \in D_0(\mathscr{B})$.*

Proof. Let $f, g \in D_u(\mathscr{B})$ be such that each $x \in X$ is a point of continuity of f or g. Suppose (P_0) is true. We want to show that $F(f, g) \in D_u(\mathscr{B})$. Let $B \in \mathscr{B}$, $p_1, p_2 \in \bar{B}$ and

$$F(f(p_1), g(p_1)) \leqslant a < b \leqslant F(f(p_2), g(p_2))$$

be given. Since $D_u(\mathscr{B}) \subset D_0(\mathscr{B})$, by ($P_0$) we have $F(f, g) \in D_0(\mathscr{B})$. There exists $p_0 \in B$ such that

$$a < F(f(p_0), g(p_0)) < b.$$

Since F is continuous, there exists $\delta > 0$ such that $a < F(s, t) < b$ whenever

$$|s - f(p_0)| < \delta \quad \text{and} \quad |t - g(p_0)| < \delta.$$

Suppose f is continuous at p_0. Then there exists $B_0 \in \mathscr{B}$ such that

$$p_0 \in B_0 \subset B \quad \text{and} \quad |f(p) - f(p_0)| < \delta \text{ for all } p \in B_0.$$

Also, since $p_0 \in B_0$ and $g \in D_u(\mathscr{B})$,

$$\operatorname{Card}(\{p \in B_0 \colon |g(p) - g(p_0)| < \delta\}) \geqslant c.$$

Clearly,

$$a < F(f(p), g(p)) < b \quad \text{for } p \in B_0 \text{ with } |g(p) - g(p_0)| < \delta.$$

Since $B_0 \subset B$, we have

$$\operatorname{Card}(\{p \in B \colon F(f(p), g(p)) \in (a, b)\}) \geqslant c.$$

The proof is completed.

COROLLARY. *Let X be a locally connected topological space, $\mathscr{B}$ a base consisting of open connected sets and satisfying (1*). Let g be continuous on X. Then $f+g \in D_u(\mathscr{B})$ if $f \in D_u(\mathscr{B})$; $fg \in D_u(\mathscr{B})$ if $f \in D_u(\mathscr{B})$, and f is bounded at each $x \in X$ where $g(x) = 0$.*

This follows from results of Misik ([5], pp. 418 and 422) and the proof of Theorem 2 above.

Theorem 2 above and Theorem 3 below, in the case where X is the real line and $\mathscr{B}$ is the collection of all open intervals, are proved in [2].

LEMMA 1. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions in $D(\mathscr{B})$ converging uniformly to f, then $f \in D_u(\mathscr{B})$.*

Proof. Let $B \in \mathscr{B}$, $x, y \in \bar{B}$ and $f(x) \leqslant a < b \leqslant f(y)$ be given. We set

$$\varepsilon = \tfrac{1}{4}(b-a) \quad \text{and} \quad \eta \in \left(\tfrac{1}{2}(a+b) - \varepsilon,\ \tfrac{1}{2}(a+b) + \varepsilon\right).$$

There exists n_0 such that $|f(z) - f_n(z)| < \varepsilon$ for all $n \geqslant n_0$ and all $z \in X$. Then $f_{n_0}(x) < \eta < f_{n_0}(y)$, and hence there exists $z_\eta \in B$ such that $f_{n_0}(z_\eta) = \eta$.

Thus

$$f(z_\eta) \in (\eta - \varepsilon, \eta + \varepsilon) \subset (a, b).$$

Clearly,

$$\{z \in B: f(z) \in (a, b)\} \supset \left\{z_\eta: \eta \in \left(\frac{a+b}{2} - \varepsilon, \frac{a+b}{2} + \varepsilon\right)\right\}$$

and

$$\operatorname{Card}(\{z \in B: f(z) \in (a, b)\}) \geqslant c.$$

Thus Lemma 1 is proved.

In the sequel, we consider the space E_2 with the base $\mathscr{B}$ consisting of all open intervals in E_2.

LEMMA 2. *Let $f \in D_u(\mathscr{B})$ and $\varepsilon > 0$ be given. Then there exists $g \in D(\mathscr{B})$ such that $|f(p) - g(p)| < \varepsilon$ for all $p \in E_2$.*

Proof. Decompose the real line R into countably many half-open intervals,

$$R = \bigcup_{n=1}^{\infty} I_n,$$

each $I_n = [a_n, b_n)$ has length $b_n - a_n < \varepsilon$. Let

$$A_n = f^{-1}(I_n^\circ) \quad \text{and} \quad \mathscr{N} = \{n: A_n \neq \emptyset\}.$$

Then $\{A_n: n \in \mathscr{N}\}$ is pairwise disjoint and, for each $n \in \mathscr{N}$, A_n is c-dense ($\mathscr{B}$) in itself. By the Proposition, there exists a pairwise disjoint sequence $\{A_{ni}\}_{i=1}^{\infty}$ such that

$$A_n = \bigcup_{i=1}^{\infty} A_{ni},$$

each A_{ni} is c-dense ($\mathscr{B}$) in A_n, and hence in itself. Let $\{J_i\}_{i=1}^{\infty}$ be an enumeration of all the open intervals with "rational end points" in E_2. Then

(#) for every $p \in E_2$ and $B \in \mathscr{B}$ such that $p \in B$ there exists i such that $p \in J_i \subset B$.

Let $P_{ni} = A_{ni} \cap J_i$ for each $n \in \mathscr{N}$ and $i = 1, 2, \ldots$ Since A_{ni} is c-dense ($\mathscr{B}$) in A_n and $J_i \in \mathscr{B}$,

$$\operatorname{Card}(P_{ni}) = c \quad \text{if } P_{ni} \neq \emptyset.$$

For each (n, i) such that $P_{ni} \neq \emptyset$, let g_{ni} be a one-to-one onto map from P_{ni} to $[a_n, b_n)$. We define g as follows:

$$g(p) = \begin{cases} g_{ni}(p) & \text{if } p \in P_{ni} \ (n \in \mathscr{N},\ i = 1, 2, \ldots), \\ f(p) & \text{otherwise.} \end{cases}$$

It should be noted that if $(n, i) \neq (n', i')$ for some $n, n' \in \mathcal{N}$ and some positive integers i, i', then $A_{ni} \cap A_{n'i'} = \emptyset$, and hence $P_{ni} \cap P_{n'i'} = \emptyset$. Thus g is well defined.

Since $b_n - a_n < \varepsilon$ for each n, $g(p) \in [a_n, b_n)$ and $f(p) \in (a_n, b_n)$ for each $p \in P_{ni} \subset A_n$, we have $|f(p) - g(p)| < \varepsilon$ for all $p \in E_2$. It remains to show that $g \in D(\mathcal{B})$.

First, we note that $g(p) \in I_n$ if and only if $f(p) \in I_n$, and if $f(p) = a_n$ for some n, then $g(p) = a_n$. In other words, if $g(p) \in I_n^\circ$, then $f(p) \in I_n^\circ$.

Let $B \in \mathcal{B}$, $p, q \in B$ and $\eta \in R$ such that $g(p) < \eta < g(q)$ be given. There exist n_1, n_2, n_3 such that $g(p) \in I_{n_1}$, $g(q) \in I_{n_2}$ and $\eta \in I_{n_3}$.

Case 1. $n_3 = n_2$. Then

$$g(q) \in I_{n_3} \quad \text{and} \quad a_{n_3} \leqslant \eta < g(q) < b_{n_3}.$$

By the above observation, $f(q) \in I_{n_3}^\circ$, that is, $q \in A_{n_3} \cap \bar{B}$. There exists $q' \in A_{n_3} \cap B$.

Case 2. $n_3 \neq n_2$. Then $I_{n_2} \cap I_{n_3} = \emptyset$. By the inequalities

$$a_{n_3} \leqslant \eta < g(q) < b_{n_2},$$

we see that $b_{n_3} \leqslant a_{n_2}$. If $f(p) \leqslant \eta$, then

$$f(p) \leqslant \eta < b_{n_3} \leqslant a_{n_2} \leqslant f(q).$$

There exists $q' \in B$ such that

$$f(q') \in (\eta, b_{n_3}) \subset I_{n_3}^\circ,$$

that is, $q' \in A_{n_3} \cap B$. If $f(p) > \eta$, then the inequalities

$$a_{n_1} \leqslant g(p) < \eta < f(p) < b_{n_1}$$

imply $\eta \in I_{n_1}$, and hence $n_1 = n_3$. Also, the same inequalities imply $p \in A_{n_1}$. Hence $p \in A_{n_3} \cap \bar{B}$. There exists $q' \in A_{n_3} \cap B$.

In any case, there is a $q' \in A_{n_3} \cap B$. By (#), there exists i such that $q' \in J_i \subset B$. Now $J_i \in \mathcal{B}$ and $q' \in A_{n_3} \cap J_i$. It follows that

$$P_{n_3 i} = A_{n_3 i} \cap J_i \neq \emptyset$$

since $A_{n_3 i}$ is c-dense $(\mathcal{B})$ in A_{n_3}. Clearly,

$$P_{n_3 i} \subset J_i \subset B, \quad g(B) \supset g(P_{n_3 i}) = I_{n_3},$$

that is, there exists $z \in B$ such that $g(z) = \eta$. Thus Lemma 2 is proved.

THEOREM 3. *$f \in D_u(\mathcal{B})$ if and only if f is the uniform limit of a sequence of functions in $D(\mathcal{B})$.*

This is a consequence of Lemmas 1 and 2.

4. The maximum of functions in $D_u(\mathscr{B})$ or $D(\mathscr{B})$.

THEOREM 4. *Let* $f \in D_u(\mathscr{B})$. *Then* $\max(f, g) \in D_u(\mathscr{B})$ *for every* $g \in D_u(\mathscr{B})$ *if and only if f is upper semi-continuous on* E_2.

Proof. It is proved in [6] that, in the space E_2, the base $\mathscr{B}$ consisting of all open intervals in E_2 fulfils conditions (1*) and (2). Thus the "if" part follows from Theorem 1. For the "only if" part, the analogous statement for $D_0(\mathscr{B})$ is proved in [6] by constructing a function $g \in D_0(\mathscr{B})$ but $\max(f, g) \notin D_0(\mathscr{B})$ in the case where f is not upper semi-continuous on E_2. It can be shown that the function g constructed there is actually in $D_u(\mathscr{B})$ when $f \in D_u(\mathscr{B})$. Now we sketch the construction and the proof here. With no loss of generality, we assume that f is bounded from below.

Suppose f is not upper semi-continuous at $p_0 = (x_0, y_0)$. Then there is a number K such that

$$f(p_0) < K < \overline{\lim_{p \to p_0}} f(p)$$

and

$$2K < f(p_0) + \overline{\lim_{p \to p_0}} f(p).$$

Let

$$p_0(\mathrm{I}) = \{p = (x, y)\colon\ x > x_0,\ y > y_0\},$$

$$p_0(\mathrm{II}) = \{p = (x, y)\colon\ x < x_0,\ y > y_0\},$$

$$p_0(\mathrm{III}) = \{p = (x, y)\colon\ x < x_0,\ y < y_0\}$$

and

$$p_0(\mathrm{IV}) = \{p = (x, y)\colon\ x > x_0,\ y < y_0\}.$$

Since $f \in D_u(\mathscr{B})$, we can show that

$$\overline{\lim_{p \to p_0}} f(p) = \max\{\overline{\lim_{\substack{p \to p_0 \\ p \in p_0(\Lambda)}}} f(p)\colon\ \Lambda = \mathrm{I, II, III, IV}\}.$$

Also, for each $\Lambda = \mathrm{I, II, III, IV}$, there exists a sequence

$$\{p_n\}_{n=1}^{\infty} \subset p_0(\Lambda)$$

such that $p_n \to p_0$ and $f(p_n) \to f(p_0)$. Let

$$X_\Lambda = \overline{p_0(\Lambda)} - \{p_0\} \quad \text{and} \quad \mathscr{B}_\Lambda = \{B \cap X_\Lambda\colon\ B \in \mathscr{B},\ B \cap X_\Lambda \neq \emptyset\}.$$

Then $\mathscr{B}_\Lambda$ is a base for X_Λ, satisfies (1*), and each member of $\mathscr{B}_\Lambda$ is connected.

In the case

$$\overline{\lim_{\substack{p \to p_0 \\ p \in p_0(\Lambda)}}} f(p) \leqslant 2K - f(p_0),$$

there is a $U_\Lambda \in \mathscr{B}$ such that

$$U_\Lambda \subset p_0(\Lambda), \quad p_0 \in \bar{U}_\Lambda \quad \text{and} \quad f(p) \leqslant 2K - f(p_0) + 1$$

for every $p \in U_\Lambda$. That is, f is also bounded from above on U_Λ. We assume that the above-mentioned sequence $\{p_n\}_{n=1}^{\infty} \subset U_\Lambda$. Let

$$A_{\Lambda 1} = \{p_n \colon n = 1, 2, \ldots\}, \quad A_{\Lambda 2} = X_\Lambda - U_\Lambda.$$

For each $p \in X_\Lambda$, we define

$$h_\Lambda(p) = \frac{d(p, A_{\Lambda 1})}{d(p, A_{\Lambda 1}) + d(p, A_{\Lambda 2})},$$

where d is the usual distance. Then h_Λ is a continuous function on X_Λ. We have

$$h_\Lambda(A_{\Lambda 1}) = 0, \quad h_\Lambda(A_{\Lambda 2}) = 1$$

and

$$h_\Lambda(p) \in (0, 1) \quad \text{if } p \in X_\Lambda - A_{\Lambda 1} - A_{\Lambda 2}.$$

By the Corollary, the function g_Λ on X_Λ defined by

$$g_\Lambda(p) = 2K h_\Lambda(p) - (2h_\Lambda(p) - 1) f(p) \quad \text{for } p \in X_\Lambda$$

is in $D_u(\mathscr{B}_\Lambda)$.

In the case

$$\overline{\lim_{\substack{p \to p_0 \\ p \in p_0(\Lambda)}}} f(p) > 2K - f(p_0),$$

we define

$$g_\Lambda(p) = 2K - f(p) \quad \text{for } p \in X_\Lambda.$$

Again, $g_\Lambda \in D_u(\mathscr{B}_\Lambda)$.

Since $g_\Lambda(p) = g_{\Lambda'}(p)$ for $p \in X_\Lambda \cap X_{\Lambda'}$, we can define g on E_2 as follows:

$$g(p) = \begin{cases} g_\Lambda(p) & \text{if } p \in X_\Lambda \ (\Lambda = \text{I, II, III, IV}), \\ f(p_0) & \text{if } p = p_0. \end{cases}$$

Using the fact that $g_\Lambda \in D_u(\mathscr{B}_\Lambda)$ and the way we define g_Λ for each Λ, we can show that $g \in D_u(\mathscr{B})$. The proof is completed.

THEOREM 5. *Let* $f \in D(\mathscr{B})$. *Then* $\max(f, g) \in D(\mathscr{B})$ *for every* $g \in D(\mathscr{B})$ *if and only if* f *is upper semi-continuous on* E_2.

This follows immediately from Theorems 1, 3 and 4.

Remark. The Proposition and Theorems 3–5 can be readily extended to the n-dimensional Euclidean space.

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