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QUADRATIC FORM SCHEMES DETERMINED BY HERMITIAN FORMS

BY

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Introduction. A *quadratic form scheme* is a triple $(g, -1, d)$, where g is an elementary 2-group, -1 is a distinguished element in g and d is a mapping assigning to each a in g a subgroup $d(a)$ of g. We require that the following four axioms hold:

C1. $a \in d(a)$ for any $a \in g$;

C2. $b \in d(a)$ implies $(-1)a \in d((-1)b)$, for any $a, b \in g$;

C3. $\bigcup \{d(x)\colon x \in ad(ab)\} = \bigcup \{ad(ax)\colon x \in d(b)\}$, for any $a, b \in g$;

C4. $\langle a_1, \ldots, a_n, c\rangle \cong \langle b_1, \ldots, b_n, c\rangle \Rightarrow \langle a_1, \ldots, a_n\rangle \cong \langle b_1, \ldots, b_n\rangle$.

Here the finite sequence $\langle a_1, \ldots, a_n\rangle$ of elements of g is said to be a *form of dimension* n and $\cong$ is the equivalence relation defined as follows: $\langle a\rangle \cong \langle b\rangle$ iff $a = b$, $\langle a_1, a_2\rangle \cong \langle b_1, b_2\rangle$ iff $a_1 a_2 = b_1 b_2$ and $b_1 \in a_1 d(a_1 a_2)$ and for $n \geqslant 3$, $\langle a_1, \ldots, a_n\rangle \cong \langle b_1, \ldots, b_n\rangle$ iff there is a finite chain of forms of dimension n beginning with $\langle a_1, \ldots, a_n\rangle$ and ending with $\langle b_1, \ldots, b_n\rangle$ such that any two neighbors in the chain have at least $n-2$ elements in common and the remaining two elements make equivalent forms of dimension 2.

The motivating example is the scheme of a field F of characteristic not two, where $g = F^{\cdot}/F^{\cdot 2}$, $-1 = (-1)F^{\cdot 2}$, $d(a)$ is the subgroup of cosets represented by the binary quadratic form $X^2 + aY^2$ over F and $\cong$ is the isometry relation of diagonalized quadratic forms over F.

It is an open question whether every quadratic form scheme comes from a field. In this paper we show that Hermitian forms over certain division algebras also determine quadratic form schemes. Whether or not these schemes are always realizable by fields is an open problem (**P 1316**) but we show that in two special cases the quadratic extension of a field on quaternion algebras the scheme determined by Hermitian forms is realizable as the factor scheme of the scheme of a field. Again, in general, it is not known whether the factor of a scheme coming from a field comes itself from a field. (**P 1317**)

1. Notation and terminology. Throughout we denote by D a division algebra with an involutory antiautomorphism σ and assume that F

$= \{x \in D\colon \sigma(x) = x\}$ is contained in the center of D. For x in D we define the norm $N(x) = x \cdot \sigma(x)$. We assume that $N(x) = N(\sigma(x))$. The norm is a group homomorphism from $D^{\cdot}$ into $F^{\cdot}$ and $F^{\cdot 2} \subset N(D^{\cdot})$. A Hermitian space is a finite dimensional left D-linear vector space V with a non-degenerate Hermitian form f. Thus $f\colon V \times V \to D$ is D-linear in the first variable and $f(x, y) = \sigma(f(y, x))$. We say f represents $a \in D$ if there is an $x \in V$ such that $f(x, x) = a$. Observe that then $\sigma(a) = a$, that is, the values represented by f belong to F and for any $d \in D$, $f(dx, dx) = aN(d)$, so that f represents cosets of $N(D^{\cdot})$ in $F^{\cdot}$.

Any Hermitian space (V, f) has an orthogonal basis $\{x_1, \ldots, x_n\}$; if $f(x_i, x_i) = a_i$, $i = 1, 2, \ldots, n$, the diagonal matrix diag $(a_1, \ldots, a_n)$ is said to be a diagonalization for (V, f). Since we are assuming f is non-degenerate, all the a_i's are different from zero.

Basic facts on Hermitian spaces can be found in [3], [4], [5], [8], and for details concerning quadratic form schemes see [1], [2] and [7].

2. The main lemma. A very well-known result on quadratic forms states that two binary quadratic forms over a field are equivalent if they represent a common element and have equal determinants (up to squares) (cf. [6], Prop. 5.1, p. 20). Our basic observation is that an analogous result holds for Hermitian spaces of dimension two.

LEMMA 2.1. *Let (V, f) and (W, h) be two-dimensional Hermitian spaces with diagonalizations* $\mathrm{diag}(a, b)$ *and* $\mathrm{diag}(c, d)$, *respectively. The spaces (V, f) and (W, h) are isometric if and only if the following two conditions are satisfied*:

(i) *$c = au + bv$ for some $u, v \in N(D)$;*

(ii) *$cd = abw$ for a certain $w \in N(D)$.*

Proof. Assume first $(V, f) = (W, h)$. We take the point of view that we have two orthogonal bases $\{x, y\}$ and $\{z, t\}$ of the same Hermitian space (V, f) with $a = f(x, x)$, $b = f(y, y)$, $c = f(z, z)$ and $d = f(t, t)$. Let $z = px + qy$ and $t = rx + sy$ for some $p, q, r, s \in D$. We have

$$c = f(z, z) = f(px, px) + f(qy, qy) = aN(p) + bN(q)$$

which proves (i), and

$$0 = f(z, t) = f(px, rx) + f(qy, sy) = pa\sigma(r) + qb\sigma(s) \tag{2.1.1}$$

and

$$cd = f(z, z) f(t, t) = \big(N(p)a + N(q)b\big)\big(N(r)a + N(s)b\big),$$

whence

$$cd = N(pr)a^2 + N(qs)b^2 + \big(N(ps) + N(qr)\big)ab. \tag{2.1.2}$$

From (2.1.1) we get $ap\sigma(r) = -bq\sigma(s)$, hence also $ar\sigma(p) = -bs\sigma(q)$ and so

$$a^2 N(pr) = b^2 N(qs). \tag{2.1.3}$$

Substituting (2.1.3) into (2.1.2) gives

$$cd = \left(N(ps) + \frac{2b}{a} N(qs) + N(qr)\right)ab. \tag{2.1.4}$$

If $s = 0$, we get $cd = N(qr)ab$, as required in (ii). If $s \neq 0$, we get from (2.1.3)

$$N(q) = \frac{a^2}{b^2} 2N(prs^{-1})$$

and substituting this into (2.1.4) gives

$$\begin{aligned} cd &= \left(N(ps) + 2\frac{a}{b}N(pr) + \left(\frac{a}{b}\right)^2 N(ps^{-1})N(r)^2\right)ab \\ &= N(ps^{-1})\left(N(s) + \frac{a}{b}N(r)\right)^2 ab \\ &= N\left(ps^{-1}\left(N(s) + \frac{a}{b}N(r)\right)\right)ab, \end{aligned}$$

as required in (ii).

Now let us assume we are given the diagonalization $\operatorname{diag}(a, b)$ for (V, f) and c, d satisfy (i) and (ii). Let $u = N(p)$, $r = N(q)$ and $\{x, y\}$ be the orthogonal base with $f(x, x) = a$, $f(y, y) = b$. Take $z = px + qy$; then $f(z, z) = aN(p) + bN(q) = c \neq 0$ (we are assuming all the spaces are non-degenerate) so that z is an anisotropic vector in the space (V, f). Pick up a vector $t' \in V$ orthogonal to z. Then $\{z, t'\}$ is an orthogonal base for (V, f) and by what has been proved above, there is an $r \in D^{\cdot}$ such that $cd' = abN(r)$, where $d' = f(t', t')$.

By (ii), $cd = abN(s)$, for a certain $s \in D^{\cdot}$. It follows $d' = dN(rs^{-1})$ and now putting $t = r^{-1}st'$ we get

$$f(t, t) = N(r^{-1}s) f(t', t') = d.$$

Thus $\{z, t\}$ is an orthogonal base for (V, f) and the corresponding diagonalization is $\operatorname{diag}(c, d)$. It follows that (V, f) and (W, h) are isometric, as required.

COROLLARY 2.2. *The non-zero values represented by Hermitian form with diagonalization* $\operatorname{diag}(1, a)$ *form a subgroup of* $F^{\cdot}$ *containing* $N(D^{\cdot})$.

Proof. First observe that for any Hermitian form f and any $a \in F^{\cdot}$ the mapping $af\colon V \times V \to D$ defined by $af(x, y) = a'f(x, y)$ provides V with a new Hermitian form. If f and g are isometric Hermitian forms, so are af and

ag, and if f has diagonalization $\mathrm{diag}(a_1, \ldots, a_n)$, af has diagonalization $\mathrm{diag}(aa_1, \ldots, aa_n)$.

Now let f be a form with diagonalization $\mathrm{diag}(1, a)$ and let f represent b. Then by Lemma 2.1

$$\langle 1, a\rangle \cong \langle b, ab\rangle,$$

where the notation $\langle c, d\rangle$ is used for the Hermitian space with diagonalization $\mathrm{diag}(c, d)$.

If f represents another element c in $F^{\bullet}$, then also $\langle 1, a\rangle \cong \langle c, ac\rangle$. Hence $\langle b, ab\rangle = \langle c, ac\rangle$ and after scaling both forms by b we get isometric forms $\langle b^2, ab^2\rangle$ and $\langle bc, abc\rangle$. Since $\langle b^2, ab^2\rangle \cong \langle 1, a\rangle$, we conclude $\langle 1, a\rangle$ represents bc. Further, if $f(x, x) = b$, $f(b^{-1}x, b^{-1}x) = b^{-1}$ and this proves the non-zero values represented by f form a subgroup of $F^{\bullet}$. Finally, if $f(x, x) = 1$, $f(dx, dx) = N(d)$, for any $d \in D^{\bullet}$, as required.

Remark 2.3. In the case of quadratic forms the corresponding result can be obtained immediately from the identity

$$(x^2+ay^2)(u^2+av^2) = (xu-ayv)^2+a(xv+yu)^2.$$

3. Schemes and Hermitian forms. We write $g = g(D, \sigma)$ for the factor group $F^{\bullet}/N(D^{\bullet})$ and -1 for coset $(-1)N(D^{\bullet})$. For any $a \in F^{\bullet}$ let $d(aN(D^{\bullet}))$ be the set of all cosets $N(D^{\bullet})$ represented by 2-dimensional Hermitian space $\langle 1, a\rangle$.

THEOREM 3.1. *$(g(D, \sigma), -1, d)$ is a quadratic form scheme.*

Proof. By Corollary 2.2, $d(aN(D^{\bullet}))$ is a subgroup of g. Certainly C1 is satisfied. Before going on let us make the following remark. For $a \in F^{\bullet}$ let $D\langle 1, a\rangle$ denote the set of elements of $F^{\bullet}$ represented by the Hermitian form $\langle 1, a\rangle$. Then $D\langle 1, a\rangle$ is a subgroup of $F^{\bullet}$ containing $N(D^{\bullet})$ (Corollary 2.2) and so it consists of cosets of $N(D^{\bullet})$ in $F^{\bullet}$ forming $d(aN(D^{\bullet}))$. Thus to determine $d(aN(D^{\bullet}))$ it is sufficient to find $D\langle 1, a\rangle$ and this is $\{N(D)+aN(D)\}\setminus\{0\}$.

We take this point of view when checking C2 and C3.

Suppose $f(x, x) = 1$, $f(y, y) = a$ and $f(x, y) = 0$. If $b \in D\langle 1, a\rangle$ then $b = N(p)+aN(q)$ for some $p, q \in D$. If $N(q) \neq 0$, then

$$-a = N\left(\frac{p}{q}\right)-bN\left(\frac{1}{q}\right) \in D\langle 1, -b\rangle,$$

as required. If $N(q) = 0$, $b = N(p)$ and $\langle 1, -b\rangle \cong \langle 1, -1\rangle$. The Hermitian space $\langle 1, -1\rangle$ represents everything represented by the hyperbolic plane over the field F, hence it represents $-a$. Thus again $-a \in D\langle 1, -b\rangle$ and C2 is satisfied.

To prove C3 we observe that

$$\begin{aligned}\bigcup\{D\langle 1, x\rangle\colon\ x\in aD\langle 1, ab\rangle\} &= \bigcup\{D\langle 1, x\rangle\colon\ x\in D\langle a, b\rangle\}\\ &= \bigcup\{N(D)+xN(D)\colon\ x\in aN(D)+bN(D)\}\setminus\{0\}\\ &= \{N(D)+aN(D)+bN(D)\}\setminus\{0\},\end{aligned}$$

and on the other hand

$$\begin{aligned}\bigcup\{aD\langle 1, ax\rangle\colon\ x\in D\langle 1, b\rangle\} &= \bigcup\{D\langle a, x\rangle\colon\ x\in D\langle 1, b\rangle\}\\ &= \{aN(D)+N(D)+bN(D)\}\setminus\{0\}.\end{aligned}$$

Thus C3 holds as a consequence of the commutativity of addition in D.

Now C4. For $a_1, \ldots, a_n \in F^{\cdot}$ we understand by $\langle a_1 N(D^{\cdot}), \ldots, a_n N(D^{\cdot})\rangle$ the class of isometric Hermitian spaces with diagonalizations $\operatorname{diag}(a_1\lambda_1, \ldots, a_n\lambda_n)$, $\lambda_i \in N(D^{\cdot})$. The relation $\cong$ is meant to be the equality relation between classes of isometric Hermitian spaces. Then $\langle aN(D^{\cdot})\rangle \cong \langle bN(D^{\cdot})\rangle$ iff $aN(D^{\cdot}) = bN(D^{\cdot})$ and for dimension 2, Lemma 2.1 assures that $\cong$ is the equivalence relation needed in C4. As to the higher dimensions, a, straightforward modification of the proof of Witt's chain-equivalence theorem for quadratic forms establishes the result for Hermitian forms (cf. [6], p. 21). Now C4 holds by the Witt cancellation theorem for Hermitian forms (cf. [3], p. 21). This proves the theorem.

Two quadratic form schemes $(g, -1, d)$ and $(g', -1', d')$ are said to be isomorphic if there exists a group isomorphism $d\colon g \to g'$ such that $\alpha(-1) = -1'$ and $\alpha(d(a)) = d'(\alpha(a))$ for every $a \in g$.

Remark 3.2. Denote by $S(D, \sigma)$ the quadratic form scheme described in the Theorem 3.1 and by $S(F)$ the quadratic form scheme coming from the field F. Then $S(F, \mathrm{id})$ is isomorphic to $S(F)$.

A scheme $S = (g, -1, d)$ is said to be realizable by a field iff there exists a field F whose quadratic form scheme is isomorphic to S. It is not known whether every quadratic form scheme is realizable by a field. Theorem 3.1 provides examples of schemes whose realizability by fields is unknown and perhaps gives a chance to find a scheme not realizable by any field. The proposition below suggests that there is no easy answer to this question even in the case of familiar algebras D.

Recall first the concept of a factor scheme (cf. [1]). Let φ be a Pfister form in the scheme $S = (g, -1, d)$ and D be its value group. Put $g' = g/D\varphi$, $-1' = (-1)D\varphi$ and $d'(aD\varphi) =$ the value group of the Pfister form $\langle 1, a\rangle \otimes \varphi$ modulo $D\varphi$. Then $S' = (g', -1', d')$ is a quadratic form scheme and S' is said to be the *factor scheme of S modulo $D\varphi$* (cf. [1]).

Even if S comes from a field F, it is an open question whether S' is realizable by a field. Now it turns out that in two typical cases the schemes supplied by Theorem 3.1 are isomorphic to the factor schemes of the scheme of a field.

PROPOSITION 3.3. (i) *Let* $D = F(\sqrt{-a})$ *be a quadratic extension of a field* F *of characteristic not* 2 *and* σ *be the non-trivial* F*-automorphism of* D. *Then the scheme* $S = (g(D, \sigma), -1, d)$ *is isomorphic to the factor scheme of the scheme* $S(F)$ *of* F *modulo* $D\langle 1, a\rangle$.

(ii) *Let* $D = \left(\frac{-a, -b}{F}\right)$ *be a quaternion algebra over a field* F *of characteristic not* 2 *and* σ *be the usual conjugation. Then* $S = (g(D, \sigma), -1, d)$ *is isomorphic to the factor scheme of the scheme* $S(F)$ *of* F *modulo* $D\langle 1, a, b, ab\rangle$.

Proof. We have $g(D, \sigma) = F^{\cdot}/N(D^{\cdot})$ so that $g(D, \sigma) = F^{\cdot}/D\langle 1, a\rangle$ in the first case and $g(D, \sigma) = F^{\cdot}/D\langle 1, a, b, ab\rangle$ in the second.

(i) The canonical isomorphism

$$F^{\cdot}/D\langle 1, a\rangle \to F^{\cdot}/F^{\cdot 2}/D\langle 1, a\rangle/F^{\cdot 2}$$

establishes the isomorphism of S and $S(F)$ modulo $D\langle 1, a\rangle$.

(ii) The canonical isomorphism

$$F^{\cdot}/D\langle 1, a, ab\rangle \to F^{\cdot}/F^{\cdot 2}/D\langle 1, a, b, ab\rangle/F^{\cdot 2}$$

establishes the isomorphism of S and $S(F)$ modulo $D\langle 1, a, b, ab\rangle$.

A final remark is concerned with the Witt rings of the objects considered above. First, for any scheme S one can define the Witt ring $W(S)$ of the scheme S by using the same approach as in the field case. On the other hand one has the Witt group $W(D, \sigma)$ of Hermitian forms over D with involution σ and tensor products make it into a ring in the case D is commutative (cf. [5]). Comparing the construction we find out that if D is commutative, there is a natural ring isomorphism $W(S(D, \sigma)) \to W(D, \sigma)$, and in the non-commutative case there is a natural group isomorphism $W(S(D, \sigma)) \to W(D, \sigma)$. Since $W(S(D, \sigma))$ is a ring, this isomorphism induces a ring structure on the additive group $W(D, \sigma)$. It is not readily seen how to explain geometrically the induced multiplication in $W(D, \sigma)$, D non-commutative.

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