COLLOQUIUM MATHEMATICUM

VOL. XXV 1972 FASC. 1

INDUCTIVE INVARIANTS OF CLOSED EXTENSIONS OF MAPPINGS

BY

TOGO NISHIURA (DETROIT, MICHIGAN)

The origin of the main result of this note* is the following conjecture of A. Lelek:

If f is a continuous mapping of a separable metrizable space X onto a separable metrizable space $f(X)$, then

$$\dim X \leqslant \dim f(X) + \max[\dim f, \operatorname{def} X] + \operatorname{loc}\operatorname{com} f + 1.$$

In [3], Lelek proved the conjecture for loc com$f = -1$ and posed the conjecture as a problem (P 469).

The conjecture is proved correct in this note. We prove a theorem concerning closed extensions of continuous maps (see Theorem 3.1). The a ove given conjecture is an immediate corollary to this theorem.

1. Preliminary definitions. We give in this section the necessary definitions and an elementary lemma needed to prove the main Theorem 3.1. All spaces under consideration are separable metrizable spaces. We agree that, for $A \subset X$, $\mathrm{Cl}_X(A)$, $\mathrm{Fr}_X(A)$ and $\mathrm{Int}_X(A)$ are the closure, boundary and interior of A in the space X, respectively.

1.1. Definition. Let X be a space and $C(X)$ be the family of compactifications of X. The *compactness deficiency* of X is the number

$$\operatorname{def} X = \min\{\dim(Y - X):\ Y \in C(X)\}.$$

1.2. Definition. Let $\mathscr{T}$ be a family of spaces which is topologically closed; i.e., $F \in \mathscr{T}$ and F' homeomorphic to F imply $F' \in \mathscr{T}$. The *inductive invariant* $I(X, \mathscr{T})$ *induced by* $\mathscr{T}$ is defined for every space X as follows:

$$I(X, \mathscr{T}) = -1 \text{ if and only if } X \in \mathscr{T}.$$

For each integer $n \geqslant 0$, $I(X, \mathscr{T}) \leqslant n$ provided that each point of X has arbitrarily small open neighborhoods U in X such that $I\big(\mathrm{Fr}_X(U)\big), \mathscr{T} \leqslant n-1$.

* This research was partially supported by the National Science Foundation grant GP 12915.

$I(X, \mathscr{T}) = n$ is defined in the obvious manner for $n = -1, 0, 1, 2, \ldots$ $\ldots, \infty$.

In this note we will let $\mathscr{T}$ be the family of locally compact spaces and denote $I(X, \mathscr{T})$ by $\operatorname{loccom} X$.

1.3. LEMMA. *Suppose $Y \supset X$ and $n \geqslant 0$. Then,* $\operatorname{loccom} X \leqslant n$ *if and only if each point of X has arbitrarily small open neighborhoods U in Y such that*

$$\operatorname{loccom}\left(\operatorname{Fr}_Y(U) \cap X\right) \leqslant n-1.$$

Proof. The lemma follows from [4], theorem 3.5, since each closed subspace of a locally compact space is again locally compact.

1.4. Notation. Let $f\colon X \to f(X)$. Then we agree that

$$\dim f = \sup\{\dim f^{-1}(w)\colon w \in f(X)\}$$

and

$$\operatorname{loccom} f = \sup\{\operatorname{loccom} f^{-1}(w)\colon w \in f(X)\}.$$

Also we agree that

$$\operatorname{loccom}(\partial f) = \sup\{\operatorname{loccom} \operatorname{Fr}_X[f^{-1}(w)]\colon w \in f(X)\}.$$

Let us remark, for closed continuous mappings f, that $\operatorname{Fr}_X[f^{-1}(w)]$ is compact for each $w \in f(X)$ and if $X_0 = \bigcup\{\operatorname{Fr}_X[f^{-1}(w)]\colon w \in f(X)\}$ and $X_0 \subset X_1 \subset X$, then $f|X_1$ is also a closed continuous mapping of X_1 onto $f(X_1)$. So, for closed mappings, $\operatorname{loccom}(\partial f) = -1$.

2. A preliminary theorem. The proof of our main theorem 3.1 relies on the following theorem which is proved by induction:

2.1. THEOREM. *Suppose $Y \supset X$. If Z is a closed subset of X and C is a closed subset of Y with $C \cap X = Z$, then*

$$\dim C \leqslant \max[\dim Z, \dim(Y-X)] + \operatorname{loccom} Z + 1.$$

The proof of the theorem is established by considering the following two statements:

Statement Δ_n. Let X, Y, C and Z be as in the hypothesis of the theorem. If $\operatorname{loccom} Z \leqslant n$ then for each $z \in Z$ and $\varepsilon > 0$ there is a subset U of C such that $z \in U$, U is open in C, diameter of $U < \varepsilon$ and

$$\dim \operatorname{Fr}_C(U) \leqslant \max[\dim Z, \dim(Y-X)] + n.$$

Statement Γ_n. Let X, Y, C and Z be as in the hypothesis of the theorem. If $\operatorname{loccom} Z \leqslant n$, then

$$\dim C \leqslant \max[\dim Z, \dim(Y-X)] + n + 1.$$

We will prove:

(i) Γ_{-1} is a true statement.

(ii) The validity of Γ_{n-1} implies the validity of Δ_n for $n \geqslant 0$.

(iii) The validity of Δ_n implies the validity of Γ_n for $n \geqslant 0$.

The above theorem follows immediately from Γ_n, $n \geqslant -1$.

2.2. PROPOSITION. *Γ_{-1} is a true statement.*

Proof. Z is locally compact. Hence Z is open in $\mathrm{Cl}_C(Z)$ and therefore each of the sets $C-\mathrm{Cl}_C(Z)$, $\mathrm{Cl}_C(Z)-Z$ and Z are F_σ subsets of C. Also $C-Z \subset Y-X$. Consequently, the sum theorem ([1], p. 30) implies $\dim C \leqslant \max[\dim Z, \dim(Y-X)]$.

2.3. PROPOSITION. *The validity of Γ_{n-1} implies the validity of Δ_n for $n \geqslant 0$.*

Proof. Let $\mathrm{loccom} Z \leqslant n$, $z \in Z$ and $\varepsilon > 0$. By 1.3, there is an open neighborhood U in C of z with diameter $U < \varepsilon$ and $\mathrm{loccom}\,[\mathrm{Fr}_C(U) \cap Z] \leqslant n-1$. Let $C' = \mathrm{Fr}_C(U)$ and $Z' = \mathrm{Fr}_C(U) \cap Z$. Then the validity of Γ_{n-1} implies $\dim C' \leqslant \max[\dim Z', \dim(Y-X)]+n$. Since $Z' \subset Z$, $\dim \mathrm{Fr}_C(U) \leqslant \max[\dim Z, \dim(Y-X)]+n$ and Δ_n is now valid.

2.4. PROPOSITION. *The validity of Δ_n implies the validity of Γ_n for $n \geqslant 0$.*

Proof. Let $\mathrm{loccom} Z \leqslant n$ and m be a positive integer. There is a countable family $\mathscr{B}_m$ of subsets U of C satisfying the following conditions:

(1) U is open in C;

(2) diameter of U is $< m^{-1}$;

(3) $\dim \mathrm{Fr}_C(U) \leqslant \max[\dim Z, \dim(Y-X)]+n$;

(4) $\bigcup \mathscr{B}_m \supset Z$.

Let $G = \bigcap \{\bigcup \mathscr{B}_m \colon m = 1, 2, \ldots\}$. Clearly, G is a G_δ subset of C containing Z and $\dim(C-G) \leqslant \dim(C-Z) \leqslant \dim(Y-X)$. Let us write $H = \bigcup_{m=1}^{\infty} [\bigcup \{\mathrm{Fr}_C(U) : U \in \mathscr{B}_m\}]$. Thus H is an F_σ subset of C and $\dim H \leqslant \max\,[\dim Z, \dim(Y-X)]+n$ by the sum theorem ([1], p. 30). Again by the sum theorem, we have

$$\dim[(C-G) \cup H] \leqslant \max[\dim Z, \dim(Y-X)]+n.$$

By [1], proposition B, p. 28,

$$\begin{aligned} \dim C &\leqslant \dim[(C-G) \cup H] + \dim(G-H) + 1 \\ &\leqslant \max[\dim Z, \dim(Y-X)] + n + 1 + \dim(G-H). \end{aligned}$$

From conditions (1) and (2) of $\mathscr{B}_m$ and the definition of H, we infer $\dim(G-H) \leqslant 0$. The proposition is completely proved.

3. Closed extension of continuous mappings. We now proceed to the statement and proof of our main theorem. The fact that closed continuous extensions of a continuous mapping exist is not surprising. The bound stated in our theorem is of prime interest.

3.1. THEOREM. *Let $f\colon X \to f(X)$ be a continuous mapping. Then there is a closed continuous mapping $F\colon V \to f(X)$ such that X is a dense subset of V, $F|X = f$ and*

$$\dim f \leqslant \dim F \leqslant \max[\dim f, \operatorname{def} X] + \operatorname{loc}\operatorname{com}(\partial f) + 1.$$

Proof. Let $Y \in C(X)$ with $\operatorname{def} X = \dim(Y - X)$. It is well known that the natural projection $p\colon Y \times f(X) \to f(X)$ is a closed mapping ([2], p. 14). Let G be the graph of f, H be the closure of G in $Y \times f(X)$ and $g = p|H$. Then g is a closed mapping of H onto $f(X)$. Let $H_0 = \bigcup \{\mathrm{Fr}_H[g^{-1}(w)]\colon w \in f(X)\}$ and $V = H_0 \cup G$. Then $g|V = F$ is a closed mapping of V onto $f(X)$. We must prove $\dim F$ satisfies the above inequality. Let $w \in f(X)$. Then

$$F^{-1}(w) = \mathrm{Fr}_H[g^{-1}(w)] \cup [f^{-1}(w) \times \{w\}].$$

It is clear that $\dim F^{-1}(w) \leqslant \max\big[\dim \mathrm{Fr}_H[g^{-1}(w)], \dim f\big]$. We compute an upper bound for $\dim \mathrm{Fr}_H\big(g^{-1}(w)\big)$. Observe that

$$Y \times \{w\} \supset g^{-1}(w), \qquad X \times \{w\} \supset f^{-1}(w) \times \{w\},$$
$$g^{-1}(w) \cap (X \times \{w\}) = f^{-1}(w) \times \{w\}.$$

Also, we have

$$\begin{aligned}\mathrm{Fr}_H[g^{-1}(w)] \cap (X \times \{w\}) &= \mathrm{Fr}_G[g^{-1}(w) \cap (X \times \{w\})] \\ &= \mathrm{Fr}_X[f^{-1}(w)] \times \{w\}.\end{aligned}$$

We can now apply theorem 2.1, with $X \times \{w\}$, $Y \times \{w\}$, $\mathrm{Fr}_H[g^{-1}(w)]$ and $\mathrm{Fr}_X[f^{-1}(w)] \times \{w\}$, to conclude that

$$\begin{aligned}\dim \mathrm{Fr}_H[g^{-1}(w)] &\leqslant \max[\dim \mathrm{Fr}_X[f^{-1}(w)], \dim(Y - X)] + \\ &+ \operatorname{loc}\operatorname{com} \mathrm{Fr}_X[f^{-1}(w)] + 1 \leqslant \max[\dim f, \operatorname{def} X] + \operatorname{loc}\operatorname{com}(\partial f) + 1.\end{aligned}$$

It now follows that $\dim F$ satisfies the required inequality and theorem is proved.

3.2. COROLLARY. *Let $f\colon X \to f(X)$ be a continuous mapping. Then*

$$\dim X \leqslant \dim f(X) + \max[\dim f, \operatorname{def} X] + \operatorname{loc}\operatorname{com} f + 1.$$

Proof. Let $F\colon V \to f(X)$ be the closed continuous mapping of theorem 3.1. Then the inequality above follows from Hurewicz's closed mapping theorem ([1], p. 91) since $\dim X \leqslant \dim V$ and $\operatorname{loc}\operatorname{com}(\partial f) \leqslant \operatorname{loc}\operatorname{com} f$. The last inequality is valid since theorem 3.3 of [4] implies

$$\operatorname{loc}\operatorname{com} \mathrm{Fr}_X[f^{-1}(w)] \leqslant \operatorname{loc}\operatorname{com} f^{-1}(w)$$

for every $w \in f(X)$.

4. Remarks. An obvious observation to make is that a by-product of the proof of theorem 3.1 is the construction of a proper mapping, i.e., a closed continuous mapping with compact point inverse, $g\colon H \to f(X)$ such that X is a dense subset of H and $g|X = f$. This leads to our final theorem.

4.1. THEOREM. *If $f\colon X \to f(X)$ is a continuous mapping, then there is a proper mapping $F\colon V \to f(X)$ such that X is a dense subset of V $F|X = f$ and*

$$\dim f \leqslant \dim F \leqslant \max[\dim f, \operatorname{def} X] + \operatorname{loccom} f + 1.$$

With much less work, one can prove the existence of proper mappings $F\colon V \to f(X)$ with X dense in V, $F|X = f$ and $\dim F \leqslant \dim f + \dim X + 1$. Since $\dim f \geqslant \operatorname{loccom} f$ and $\dim X \geqslant \operatorname{def} X \geqslant \operatorname{loccom} X \geqslant \operatorname{loccom} f$ (see [3], section 2), the bounds found in theorems 3.1 and 4.1 are sharper. To show that the bounds are in fact sharper, we have the following example. The example also shows that the bounds are in a sense best possible.

4.2. Example Let $X = A \cup B \cup C$, where $A = [0,1] \times [0,1] \times [0,1)$, $B = [0,1] \times \{0,1\} \times \{1\} \cup \{0,1\} \times [0,1] \times \{1\}$ and C is a countable dense subset of $[0,1] \times [0,1] \times \{1\}$. It is known that $\operatorname{def}(A \cup B) = 2$ ([5], theorem 4.1.1). Hence, it is easily shown that $\operatorname{def} X = 1$.

Let $f\colon X \to W$ be a continuous mapping of X onto $W = \{w \in R^3 \colon |w| \leqslant 1\}$ such that $f(B \cup C) = (0,0,1)$ and $f|A$ is a homeomorphism onto $W - \{(0,0,1)\}$. We have $\operatorname{Fr}_X[f^{-1}(w)] = f^{-1}(w)$ for every $w \in W$. Hence we conclude $\operatorname{loccom}(\partial f) = 0$.

Let $F\colon V \to W$ be any closed continuous mapping which extends $f\colon X \to W$. Let $V_0 = \bigcup \{\operatorname{Fr}_V[F^{-1}(w)] \colon w \in W\}$ and $F_0 = F|V_0$. Then F_0 is a closed continuous mapping of V_0 onto W and $F_0^{-1}(w)$ is compact for each $w \in W$; i.e., F_0 is a proper mapping of V_0 onto the compact space W. Hence V_0 is compact. X being dense in V_0, V_0 is a compactification of $A \cup B$. Therefore,

$$2 = \operatorname{def}(A \cup B) \leqslant \dim[V_0 - (A \cup B)] \leqslant \dim[V_0 - A] = \dim[F_0^{-1}(0,0,1)],$$

since $f|A$ is a homeomorphism. We conclude that

$$\dim F \geqslant \dim F_0 \geqslant 2 = \max[\dim f, \operatorname{def} X] + \operatorname{loccom}(\partial f) + 1$$

and

$$\dim f + \dim X + 1 = 5 > \dim f + \operatorname{def} X + 1 = 3.$$

REFERENCES

[1] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton 1948.
[2] K. Kuratowski, *Topology*, Vol. II, NewYork 1968.

[3] A. Lelek, *Dimension and mappings of spaces with finite deficiency*, Colloquium Mathematicum 12 (1964), p. 221-227.

[4] T. Nishiura, *Inductive invariants and dimension theory*, Fundamenta Mathematicae 59 (1966), p. 243-262.

[5] J. de Groot and T. Nishiura, *Inductive compactness as a generalization of semicompactness*, ibidem 58 (1966), p. 201-218.

DEPARTMENT OF MATHEMATICS
WAYNE STATE UNIVERSITY
DETROIT, MICHIGAN 48202

Reçu par la Rédaction le 8. 9. 1970