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# ON THE FIXED POINT FORMULA OF ATIYAH AND BOTT

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In this paper we give a simple proof of the fixed point formula of Atiyah and Bott. Let $E = \{E_i, d_i\}$ be an elliptic complex of differential operators on a closed manifold $X$. We have a sequence

$$0 \to \Gamma(E_0) \xrightarrow{d_0} \Gamma(E_1) \xrightarrow{d_1} \ldots \xrightarrow{d_{N-1}} \Gamma(E_N) \to 0.$$

We consider a differentiable mapping $f\colon X \to X$ together with a chain map $f^{\#}\colon \Gamma(E) \to \Gamma(E)$, induced by given bundle maps $\varphi_i\colon f^*E_i \to E_i$ as follows:

$f^{\#}s(x) = \varphi_i(x)s\big(f(x)\big)$ for every section $s$ of $E_i$. The maps $f^{\#}$ induce linear homomorphisms of the homology $f_i^*\colon H_i(E) \to H_i(E)$, and the Lefschetz number $L(f, E)$ is defined as

$$L(f, E) = \sum_{i=0}^{N} (-1)^i \operatorname{tr} f_i^*.$$

For a simple fixed point $x$ of $f$ the number

$$\nu(x) = \sum_{i=0}^{N} (-1)^i \frac{\operatorname{tr}\varphi_i(x)}{|\det(1-df_x)|}$$

is called the *fixed point index* of $f$ in $x$. Now we can state the fixed point formula of Atiyah and Bott (cf. [1]-[3]).

THEOREM. *Let $f\colon X \to X$ be a differentiable mapping with only simple fixed points and with a chain map $f^{\#}$ defined as above. Then the Lefschetz number $L(f, E)$ is given by the sum of all fixed point indices*: $L(f, E) = \nu(x)$.

In our proof we work with parametrics which are especially nice near the fixed points. We start with an arbitrary parametric $\{K_i, S_i\}$ of $E$. That means, we have

$$1 - S_i = d_{i-1}K_{i-1} + K_i d_i$$

on every $\Gamma(E_i)$, where $S_i\colon \Gamma(E_i) \to \Gamma(E_i)$ is a smoothing operator and $K_i\colon \Gamma(E_{i+1}) \to \Gamma(E_i)$ is a pseudo-differential operator with $\operatorname{order}(K_i) = -\operatorname{order}(d_i)$. (For the formal reason we set $K_{-1} = 0$, $K_N = 0$, $d_{-1} = 0$, and $d_N = 0$). As usual, we denote the geometric dual $E_i \otimes \Omega(X)$ of $E_i$ by $E_i'$, and the external tensor product of $E_i$ and $E_i'$ over $X \times X$ by $E_i \boxtimes E_i'$. On the diagonal $\Delta$ we can take the trace of a section of $E_i \boxtimes E_i'$ to obtain a volume form of $X$ (which is here identified with $\Delta$).

LEMMA 1. *Let $s_i \in \Gamma(E_i \boxtimes E_i')$ be the kernel of $S_i$. Then*

$$L(f, E) = \sum_{i=0}^{N} (-1)^i \int_X \operatorname{tr} \varphi_i(x) s_i(fx, x). \tag{1}$$

Proof. By composition of $f^{\#}$ and $S = \{S_i\}$ we obtain an endomorphism $T = \{T_i\}$ of $E$ with smooth kernels $t_i(x, y) = \varphi_i(x) s_i(fx, y)$. From the given chain homotopy between $S$ and the identity, we infer that $T$ and $f^{\#}$ induce the same homomorphisms of the homology and (1) follows from the alternating sum formula for the traces of smooth endomorphisms. (Cf. [1], Proposition 2.4. At the end of the paper we give a modified version which is independent of [1].)

To construct a family of parametrics from $\{K_i, S_i\}$, we choose a smooth family of real functions on $X \times X$ with the following properties. For every positive real $t$ we have a function $\varphi^t$ such that

1. every $\varphi^t$ is equal to 1 on a neighbourhood of the diagonal,
2. if $t \to 0$, then the supports of $\varphi^t$ contract on $\Delta$.

Let $k_i$ be the Schwartz kernel of $K_i$ and let $K_i^t$ be the pseudo-differential operator with kernel $k_i \varphi^t$. The operator $K_i^t$ differs from $K_i$ only by a smoothing operator. We have

$$1 - S_i^t = d_{i-1} K_{i-1}^t + K_i^t d_i, \tag{2}$$

where $S_i^t$ is a smoothing operator given by

$$S_i^t = S_i + d_{i-1}(K_{i-1} - K_{i-1}^t) + (K_i - K_i^t) d_i.$$

If we denote the kernels of $S_i^t$ and $S_i$ by $s_i^t$ and $s_i$, respectively, then

$$s_i^t = s_i + d_{i-1}(x)\big(k_i(1-\varphi^t)\big) + d_i'(y)\big(k_i(1-\varphi^t)\big). \tag{3}$$

Here and in the following the letters $x, y$ indicate that the differential operator is applied with respect to the first or the second variable, and $d_i'$ denotes the transpose of $d_i$ with respect to the natural pairing of $\Gamma(E_i)$ and $\Gamma(E_i')$. For the parametric $\{K_i^t, S_i^t\}$ we obtain, by Lemma 1, the following integral formula for the Lefschetz number:

$$L(f, E) = \sum_{i=0}^{N} \int_X \operatorname{tr} \varphi_i(x) s_i^t(fx, x). \tag{4}$$

From now on we assume that the fixed points are isolated. We can separate the fixed points $p_1, \ldots, p_k$ by open neighbourhoods $U_1, \ldots, U_k$. Let $t$ be so small that the intersection of $\operatorname{supp}(s_i^t)$ with the graph $f \times 1(X)$ of $f$ is included in the union of all $f \times 1(U_j)$. (This is possible since $\operatorname{supp}(s_i^t) \subset \operatorname{supp}(\varphi^t)$.) Then we obtain

$$L(f, E) = \sum_{j=1}^{k} \int_{U_j} \sum_{i=0}^{N} (-1)^i \operatorname{tr} \varphi_i(x) s_i^t(fx, x).$$

In the following we shall show that for suitable $\varphi^t$ the integrals on the right-hand side converge to $\nu(p_j)$ and the fixed point formula will hold (in fact, one can show that the integrals do not depend on the choice of $\varphi^t$ and, therefore, they give the exact value of the fixed point index). Now, the problem is reduced to a local one and up to the end of the proof we shall work in a neighbourhood $U = U_j$ of one of the fixed points $p = p_j$. We can assume that $U$ is contained in a coordinate patch and that the bundles $E_i$ are trivial over $U$. We have $E_i | U = U \times C^{m_i}$. All sections are to identify with their coordinates. For example, $s_i(x, y)$ is to identify with $\bar{s}_i(x, y) \otimes \otimes dy$, where $\bar{s}_i(x, y)$ denotes an $(m_i \times m_i)$-matrix depending on $x$ and $y$, and $dy$ denotes the Lebesgue volume element of the coordinate space. For simplicity, we write again $s_i(x, y)$ for $\bar{s}_i(x, y)$.

We consider in detail only the case where the operators $d_i$ are of the same order 1 (cf. Remark 1 below for the general case). Then $K_i$ are pseudo-differential operators of order $-1$. By the theory of pseudo-differential operators, we can consider $K_i$ as usual (singular) integral operators. The kernels are smooth outside the diagonal, and near the diagonal we have (with $r(x, y) = |x-y|$)

$$k_i(x, y) = O(r^{1-n}), \tag{5}$$

$$\left(\frac{\partial}{\partial x^a} + \frac{\partial}{\partial y^a}\right) k_i(x, y) = O(r^{1-n}) \tag{6}$$

or, equivalently to (6),

$$k_i(x+z, y+z) - k_i(x, y) = O(|z| r^{1-n}) \tag{7}$$

(cf., e.g., [3] for a very clear exposition). To specify $\varphi^t$ we choose a real function $h$ on $R$ such that $h([0, 1/4]) = \{1\}$ and $h\big([1/2, \infty)\big) = \{0\}$ and assume that $\varphi^t$ is given in $U \times U$ by $\varphi^t(x, y) = h(rt^{-1})$. Then we have

$$\frac{\partial}{\partial x^a} \varphi^t = O(t^{-1}) \tag{8}$$

and

$$\left(\frac{\partial}{\partial x^a}+\frac{\partial}{\partial y^a}\right)\varphi^t = 0. \tag{9}$$

Under these assumptions we can prove

LEMMA 2. *Let $a$ be a section of $U\times C^{m_i}$. Then*

$$\lim_{t\to 0}\int_U s_i^t(p,y)a(y)dy = a(p) \tag{10}$$

*and*

$$\lim_{t\to 0}\int_U s_i^t(fy,y)a(y)dy = \frac{a(p)}{|\det(1-df_p)|}. \tag{11}$$

First we point out that the Theorem follows already from (11). Let $a_0$ be an element of $C^{m_i}$. We apply (11) to the section $a(y) = \varphi_i(y)a_0$ and we obtain

$$\lim_{t\to 0}\int_U s_i^t(fy,y)\varphi_i(y)a_0dy = \frac{\varphi_i(p)a_0}{|\det(1-df_p)|}.$$

Since the formula holds for every $a_0$, we have

$$\lim_{t\to 0}\int_U s_i^t(fy,y)\varphi_i(y)dy = \frac{\varphi_i(p)}{|\det(1-df_p)|}.$$

Taking the matrix trace on both sides and using the known formula $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for matrices, by alternating the summation, we obtain the desired formula

$$\lim_{t\to 0}\int_U\sum_{i=0}^{N}(-1)^i\mathrm{tr}\varphi_i(y)s_i^t(fy,y)dy = \nu(p).$$

Proof of (10). We apply both sides of (2) to the section $a$ and take the value at $p$ ($a$ is extended as the zero section over $X$ but for $t$ being sufficiently small the relevant sections vanish outside $U$). For small $t$ we obtain

$$a(p)-\int_U s_i^t(p,y)a(y)dy = d_{i-1}(x)\int_U k_{i-1}^t(x,y)a(y)dy|_{x=p}+$$
$$+\int_U k_i^t(p,y)d_ia(y)dy.$$

We show that the integrals on the right-hand side tend to zero with $t$. For the second integral we have

$$\left|\int k_i^t(p, y) d_i a(y) dy\right| \leqslant \text{const} \int |k_i(p, y)| dy, \qquad |p-y| \leqslant t$$

(note that $k_i^t = \varphi^t k_i$ and $\varphi^t(p, y) = 0$ for $|y-p| > t$). From (5) it follows that $k_i(p, y)$ is integrable in $y$ and, therefore, this integral tends to zero with $t$. To deal with the first integral we introduce the local expression for the operator $d_{i-1}$:

$$d_{i-1} = \sum_a v_{i-1}^a \frac{\partial}{\partial x^a} + w_{i-1}.$$

As above we show that

$$\lim_{t\to 0} \int w_{i-1}(p) k_{i-1}^t(p, y) a(y) dy = 0.$$

Moreover, the summands involving the first order terms tend to zero by the following calculation using (6) and (9):

$$\frac{\partial}{\partial x^a} \int k_{i-1}^t(x, y) a(y) dy = \frac{\partial}{\partial x^a} \int k_{i-1}^t(x, x+z) a(x+z) dz$$

$$= \left(\frac{\partial}{\partial x^a} + \frac{\partial}{\partial y^a}\right) k_{i-1}^t(x, y) a(y) dy = \int \varphi^t(x, y) \left(\frac{\partial}{\partial x^a} + \frac{\partial}{\partial y^a}\right) k_{i-1}(x, y) a(y) dy$$

After the substitution $y = x+z$ it is possible to differentiate the integrand. The last integral tends to zero by the same arguments as above.

Proof of (11). We apply (10) to the section $a(y) |\det(1-df_y)|^{-1}$. and substitute $y = z-f(z)$. Then we obtain

$$\lim_{t\to 0} \int s_i^t(p, z-fz) a(z-fz) dz = \frac{a(p)}{|\det(1-df_p)|}.$$

We show that the above integral differs from $\int s_i^t(fy, y) a(y) dy$ only by $O(t)$. To do this we multiply (3) by the function $\psi^t = \varphi^{2t}$. We take $(fy, y)$ and $y$ as the arguments and integrate over $U$ to obtain

$$(12) \qquad \int s_i^t a dy = \int \psi^t s_i a dy - \int \psi^t d_{i-1}(x) k_{i-1}(1-\varphi^t) a dy - \\ - \int k_i (1-\varphi^t) d_i(a\psi^t) dy.$$

Here and in the following we omit the arguments in the integrands. Since $p$ is assumed to be a simple fixed point, there is a constant $c$ such that $|fy-y| \geqslant c|y-p|$ near $p$ and $\psi^t(fy, y)$ vanishes for $|y-p| \geqslant t/c$. The first integral on the right-hand side tends evidently to zero with $t$. To handle with the third integral on the right-hand side we note that

$k_i(fy, y) = O(|y-p|^{1-n})$ holds because of (5) and the simplicity of the fixed point. Therefore, $k_i(fy, y)$ is integrable and, using the local expression for $d_i$, we obtain

$$\int k_i(1-\varphi^t)d_i(a\psi^t)dy = \int k_i d_i(a\psi^t)dy + O(t)$$

$$= k_i\left(v_i^a \frac{\partial}{\partial x^a} + w^a\right)(a\psi^t)dy + O(t) = \sum_a \int k_i v_i^a a \frac{\partial}{\partial x^a}\psi^t dy + O(t).$$

By partial integration, using (9) and the main theorem of analysis, for the second integral on the right-hand side of (12) we get

$$\int \psi^t d_{i-1}(x)k_i(1-\varphi^t)ady = -\sum_a \int \frac{\partial}{\partial x^a}\psi^t v_{i-1}^a k_{i-1}(1-\varphi^t)ady +$$

$$+\int \psi^t(1-\varphi^t)\left[\sum_a k_i v_{i-1}^a \frac{\partial}{\partial x^a} a + w_{i-1}a + \sum_a \left(\frac{\partial}{\partial x^a} + \frac{\partial}{\partial y^a}\right)v_{i-1}^a k_i a\right]dy.$$

The term in the square brackets is integrable. Therefore, the last integral tends to zero and we obtain

$$\int s_i^t(fy, y)a(y)dy = \sum_a \frac{\partial}{\partial x^a}\psi^t(v_{i-1}^a k_{i-1} - v_i^a k_i)ady + O(t).$$

In the same way we obtain an analogous expression for

$$\int s_i^t(p, y-fy)a(y-fy)dy.$$

(We have only to replace the arguments in the integrands.) But the difference of the two expressions tends to zero with $t$, which is easily shown by using (7) and (8).

Remark 1. If the operator $d_i$ is of order $k_i$, then we have to replace 1 by $k_i$ in (5)-(7) and the partial integrations are to iterate.

Remark 2. The alternating sum formula for the traces of smooth operators is quite elementary. In our view this formula seems to be a natural starting point in a proof of the fixed point formula. But to make the paper self-contained we give an alternative proof of (4) in a special case which is sufficient for our purpose.

Introducing hermitian metrics on the bundles we can define adjoints and laplacians. By standard Hodge theory we obtain a parametric $\{K_i, S_i\}$, where the smoothing operators $S_i$ are just the harmonic projections. Here formula (1) follows easily from the fact that the harmonic sections represent the homology of the complex. But (4) is a consequence of (1) and (3). To see this we rewrite (3) in the form

$$s_i = s_i^t - d_{i-1}(x)\, q_{i-1}^t - d_i'(y)\, q_i^t. \tag{13}$$

Note that $q_i^t = k_i(1-\varphi^t)$ is smooth on $X \times X$. Now we can argue (as Kotake proved his Lemma 3) as follows. We replace $s_i$ in (1) by the right-hand side of (13). The terms involving $q_i^t$ cancel out in the alternating sum because of the identity

$$\int \operatorname{tr} f^{\#}(x)\, d_i(x)\, q(x, y)|_{x=y} = \int \operatorname{tr} f^{\#}(x)\, d_i'(y)\, q(x, y)|_{x=y},$$

which is clear for decomposing $q(x, y) = a(x) \otimes b(y)$, and by a density argument for every section in $\Gamma(E_i \boxtimes E_i')$. Thus (4) is proved.

## *REFERENCES*


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