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CONCERNING DUAL SYSTEMS OF LINEAR RELATIONS (I)

BY

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In this paper we prove two general theorems on dual systems of linear relations. As corollaries to them, we obtain many known theorems on such systems that are used in the theory of linear economic models [1].

To our purpose, it is convenient to use a modification of the matrix notation for vectors, which consists in that we give relations between vectors-rows and relations between vectors-columns an additional logical meaning. This symbolism not only leads to a compact formulation of theorems but also makes the proofs more clear. In the proof of our general theorems we use the fixed point theorem of von Neumann [3] in the formulation of Kakutani [2]:

Let K and L be bounded, closed and convex sets in Euclidean spaces R^m and R^n, respectively. Consider their Cartesian product $K\times L$ in the space R^{m+n}. Let U and V be two closed subsets of $K\times L$ such that, for every $x_0\in K$, the set U_{x_0} composed of points $y\in L$ with $(x_0, y)\in U$, and, for every $y_0\in L$, the set V_{y_0} composed of points $x\in K$ with $(x, y_0)\in V$ are non-empty, closed and convex. Then U and V have a point in common.

Notation. Lower-case Greek letters $\alpha, \beta, \varphi, \psi, \varrho, \delta$ and χ denote real numbers. Lower-case letters p, q, r, s, u, v, x and y denote column vectors with a finite number of real components, while letters a, b, c, d, e, f and g stand for real matrices. The number of rows and columns of matrices entering formulas is assumed to fulfil usual requirements of compatibility as regards matrix multiplication and addition. The transpose of a matrix will be denoted by a prime. Thus p', q', r', s', u', v', x' and y' denote always row vectors. By 0 we denote, as the need will be, a null matrix, or a column of zero's, or a row of zero's.

The conjunction of relations will be expressed by writing the relations one above the other. The alternative of relations will be expressed by writing them one after the other, with a comma between. In this spirit, inequality

$$p > q$$

will be understood as the conjunction of all the inequalities between the corresponding components of vectors p and q, with the sign $>$ between them, and

$$p' > q'$$

will be understood as the alternative of the inequalities between the corresponding components of vectors p and q, with the sign $>$ between them. An analogous meaning is given to the following relations:

$$p \geqslant q; \quad p = q; \quad p \neq q;$$
$$p' \geqslant q'; \quad p' = q'; \quad p' \neq q'.$$

Sentences will be denoted by capital letters $A, B, \ldots$ The conjunction of A and B will be denoted by $A \frown B$, the alternative by $A \smile B$. If A implies B, we write $A \subset B$. By $A \equiv B$ the equivalence of A and B will be meant. A' stands for the negation of A. Note that in this spelling de Morgan's laws for sentences concerning relations between components of vectors obtain a very compact expression, as for instance

$$(r \geqslant s)' \equiv (r' < s') \quad \text{or} \quad (r = s)' \equiv (r' \neq s').$$

The disjunction of A and B will be symbolized by $A \mid B$. The universal and the existential quantifiers will be denoted by $\bigcap$ and $\bigcup$, respectively.

K and L stand for sets in Euclidean spaces R^m and R^n, respectively. Their points will be identified with vectors ($=$ one-column matrices). When not stated otherwise explicitely, x will be supposed to run over R^m and y to run over R^n.

LEMMA. *If K and L are bounded, closed and convex sets, $\varphi(x, y)$ and $\psi(x, y)$ are continuous real functions defined for $x \in K$ and $y \in L$ such that $\varphi(x, y)$ is a convex function of y for every $x \in K$, and $\psi(x, y)$ is a convex function of x for every $y \in L$, then*

$$\bigcap_{x \in K} \bigcup_{y \in L} \big(\varphi(x, y) \leqslant \varphi\big) \frown \bigcap_{y \in L} \bigcup_{x \in K} \big(\psi(x, y) \leqslant \psi\big) \subset \bigcup_{\substack{x \in K \\ y \in L}} \begin{pmatrix} \varphi(x, y) \leqslant \varphi \\ \psi(x, y) \leqslant \psi \end{pmatrix}.$$

Proof. Note that if $\chi(y)$ is a convex function of y, then the set of y for which $\chi(y) \leqslant \chi$ is convex, and if $\chi(y)$ is continuous, then this set is closed. If we apply this to functions $\varphi(x, y)$ for a fixed x and to functions $\psi(x, y)$ for a fixed y, and use the boundedness of K and L, we see that the conditions of von Neumann's fixed point theorem are satisfied and the Lemma follows.

Theorems on linear relations. We are going to prove

THEOREM 1. *We have*

$$\bigcap_{a,b,c,d,r,s} \bigcup_{x,y} \begin{pmatrix} ax+by \geqslant r \\ cx+dy = s \\ x \geqslant 0 \end{pmatrix} \Big| \bigcup_{u,v} \begin{pmatrix} a'u+c'v \leqslant 0 \\ b'u+d'v = 0 \\ u \geqslant 0 \\ r'u+s'v > 0 \end{pmatrix}.$$

Proof. Denote the propositions

$$\bigcup_{x,y} \begin{pmatrix} ax+by \geqslant r \\ cx+dy = s \\ x \geqslant 0 \end{pmatrix} \quad \text{and} \quad \bigcup_{u,v} \begin{pmatrix} a'u+c'v \leqslant 0 \\ b'u+d'v = 0 \\ u \geqslant 0 \\ v'u+s'v > 0 \end{pmatrix}$$

by A and B. Theorem 1 may be then expressed in the form

$$\bigcap_{a,b,c,d,r,s} A \mid B$$

and is equivalent with the conjunction

$$\text{(1)} \qquad [\bigcap_{a,b,c,d,r,s} (A \cap B)'] \cap [\bigcap_{a,b,c,d,r,s} (A' \cap B')'].$$

We first prove

$$\text{(2)} \qquad \bigcap_{a,b,c,d,r,} (A \cap B)'.$$

By multiplying the vectors in A from the left by arbitrary vectors $u \geqslant 0$ and v, we obtain the implication

$$A \subset \bigcup_{\substack{x,y \\ x \geqslant 0}} \bigcap_{\substack{u,v \\ u \geqslant 0}} (u'ax+u'by+v'cx+v'dy \geqslant u'r+v's).$$

Similarly, by multiplying the vectors in B by arbitrary vectors $x \geqslant 0$ and y, we conclude that

$$B \subset \bigcup_{\substack{u,v \\ u \geqslant 0}} \bigcap_{\substack{x,y \\ x \geqslant 0}} (x'a'u+x'c'v+y'b'u+y'd'v \leqslant 0 < r'u+s'v).$$

These consequences of A and B contradict each other. So (2) holds true.

It remains to prove that

$$\text{(3)} \qquad \bigcap_{a,b,c,d,r,s} (A' \cap B')'.$$

To do this observe that

$$(4)\quad B' \equiv \bigcap_{\substack{u,v\\ u\geqslant 0}} (u'a+v'c>0,\ u'b+v'd\neq 0,\ u'r+v's\leqslant 0)$$

$$\subset \bigcap_{\substack{u,v\\ u\geqslant 0}} \bigcup_{\substack{p,q\\ p\geqslant 0}} (u'ap+v'cp+u'bq+v'dq>0,\ -u'r-v's\geqslant 0)$$

$$\subset \bigcap_{\substack{u,v\\ u\geqslant 0}} \bigcup_{\substack{p,q\\ p\geqslant 0}} \bigcup_{\substack{a\geqslant 0\\ \beta>0}} [(u'ap+v'cp+u'bq+v'dq)a-(u'r+v's)\beta\geqslant 0]$$

$$\subset \bigcap_{\substack{u,v\\ u\geqslant 0\\ u'u\leqslant 1\\ v'v\leqslant 1}} \bigcup_{\substack{x,y\\ x\geqslant 0\\ x'x\leqslant 1\\ y'y\leqslant 1}} (u'ax+v'cx+u'by+v'dy-u'r-v's\geqslant 0).$$

The first of the above implications is valid, for we can choose vector p so that only those its components are positive which correspond to positive components of vector $u'a+v'c$. Similarly, we can choose non-zero components of q and their signs to correspond to the non-zero components (if any) of $u'b+v'd$ so as to ensure the positivity of the respective products. The next implication is obvious. Finally, we obtain the last by substituting

$$x = p\frac{a}{\beta}\delta \quad\text{and}\quad y = q\frac{a}{\beta}\delta$$

with a sufficiently small positive δ.

In a similar manner we can write the following chain of implications:

$$(5)\quad A' \equiv \bigcap_{\substack{x,y\\ x\geqslant 0}} (x'a'+y'b'<r',\ x'c'+y'd'=s')$$

$$\subset \bigcap_{\substack{x,y\\ x\geqslant 0}} \bigcup_{\substack{p,q,a\\ p\geqslant 0\\ a<0}} (x'a'p+y'b'p+x'c'q+y'd'q-r'p-s'q=a)$$

$$\subset \bigcap_{\substack{x,y\\ x\geqslant 0\\ x'x\leqslant 1\\ y'y\leqslant 1}} \bigcup_{\substack{u,v,\psi\\ u\geqslant 0\\ u'u\leqslant 1\\ v'v\leqslant 1\\ \psi<0}} (u'ax+v'cx+u'by+v'dy-u'r-v's=\psi)$$

$$\subset \bigcup_{\varrho<0} \bigcap_{\substack{x,y\\ x\geqslant 0\\ x'x\leqslant 1\\ y'y\leqslant 1}} \bigcup_{\substack{u,v\\ u\geqslant 0\\ u'u\leqslant 1\\ v'v\leqslant 1}} (u'ax+v'cx+u'by+v'dy-u'r-v's\leqslant\varrho)$$

To see the last implication, take

$$\varrho = \max_{\substack{x,y\\ x\geqslant 0\\ x'x\leqslant 1\\ y'y\leqslant 1}} \min_{\substack{u,v\\ u\geqslant 0\\ u'u\leqslant 1\\ v'v\leqslant 1}} (u'ax+v'cx+u'by+v'dy-u'r-v's).$$

This quantity exists and is negative, because the function

$$\varphi\left(\begin{pmatrix}u\\v\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = u'ax + v'cx + u'by + v'dy - u'r - v's$$

is continuous and the domains considered are closed and bounded. We claim that the last conclusion of (4) and the last conclusion of (5) contradict each other. Namely, by taking the set of vectors $\begin{pmatrix}u\\v\end{pmatrix}$ with $u \geqslant 0$, $u'u \leqslant 1$ and $v'v \leqslant 1$ for K in the Lemma, the set of vectors $\begin{pmatrix}x\\y\end{pmatrix}$ with $x \geqslant 0$, $x'x \leqslant 1$ and $y'y \leqslant 1$ for L, and putting

$$-\varphi\left(\begin{pmatrix}u\\v\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = \psi\left(\begin{pmatrix}u\\v\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right),$$

we conclude that

$$\bigcup_{\substack{\binom{u}{v}\in K \\ \binom{x}{y}\in L}} \begin{pmatrix} \psi\left(\begin{pmatrix}u\\v\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) \geqslant 0 \\ \psi\left(\begin{pmatrix}u\\v\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) \leqslant \varrho < 0 \end{pmatrix}.$$

This is, however, impossible. Thus (3) holds true and the proof of Theorem 1 is complete.

Corollary 1. *If we assume, in Theorem* 1, *that only one of the matrices* a, b, c, d *is not zero, we get the four known* ([1], *p.* 41-47) *theorems on systems of non-homogeneous linear equations and inequalities. Here they are*:

$$\bigcap_{a,r} \bigcup_{x} \begin{pmatrix} ax \geqslant r \\ x \geqslant 0 \end{pmatrix} \mid \bigcup_{u} \begin{pmatrix} a'u \leqslant 0 \\ u \geqslant 0 \\ r'u > 0 \end{pmatrix} \qquad \text{([1], p. 47)},$$

$$\bigcap_{b,r} \bigcup_{y} (by \geqslant r) \mid \bigcup_{u} \begin{pmatrix} b'u = 0 \\ u \geqslant 0 \\ r'u > 0 \end{pmatrix} \qquad \text{([1], p. 46)},$$

$$\bigcap_{c,s} \bigcup_{x} \begin{pmatrix} cx = s \\ x \geqslant 0 \end{pmatrix} \mid \bigcup_{v} \begin{pmatrix} c'v \leqslant 0 \\ s'v > 0 \end{pmatrix} \qquad \text{([1], p. 44)},$$

$$\bigcap_{d,s} \bigcup_{y} (dy = s) \mid \bigcup_{v} \begin{pmatrix} d'v = 0 \\ s'v > 0 \end{pmatrix} \qquad \text{([1], p. 41)}.$$

Let us now proceed to

THEOREM 2. *We have*

$$\bigcap_{\substack{e,f,g\\ -f=f'\\ -g=g'}} \bigcup_{\substack{x,y\\ x\geqslant 0}} \begin{pmatrix} fx-ey \geqslant 0 \\ fx-ey+x > 0 \\ e'x-gy = 0 \end{pmatrix}.$$

Proof. If we substitute $\begin{pmatrix} f \\ f+i \end{pmatrix}$ for a, $\begin{pmatrix} -e \\ -e \end{pmatrix}$ for b, e' for c, $-g$ for d, x for x, y for y, $\begin{pmatrix} 0 \\ r \end{pmatrix}$ for r, $\begin{pmatrix} u \\ w \end{pmatrix}$ for u, v for v and 0 for s in Theorem 1, where by i we mean the unit matrix, we obtain proposition

$$\bigcap_{\substack{e,f,g,r\\ -f=f'\\ -g=g'\\ r>0}} \bigcup_{\substack{x,y\\ x\geqslant 0}} \begin{pmatrix} fx-ey \geqslant 0 \\ fx-ey+x \geqslant r \\ e'x-gy = 0 \end{pmatrix} \mid \bigcup_{\substack{u,v,w\\ u\geqslant 0\\ w\geqslant 0}} \begin{pmatrix} f'(u+w)+w-ev \leqslant 0 \\ -e'(u+w)-g'v = 0 \\ r'w > 0 \end{pmatrix}. \tag{6}$$

Suppose now, to the contrary, that Theorem 2 is not true. The more so we had

$$\bigcup_{\substack{e,f,g,r\\ -f=f'\\ -g=g'\\ r>0}} \bigcap_{\substack{x,y\\ x\geqslant 0}} \begin{pmatrix} fx-ey \geqslant 0 \\ fx-ey+x \geqslant r \\ e'x-gy = 0 \end{pmatrix}'.$$

In view of (6) we then had

$$\bigcup_{\substack{e,f,g,r\\ -f=f'\\ -g=g'\\ r>0}} \bigcup_{\substack{u,v,w\\ u\geqslant 0\\ w\geqslant 0}} \begin{pmatrix} f'(u+w)+w+ev \leqslant 0 \\ -e'(u+w)-g'v = 0 \\ r'w > 0 \end{pmatrix}.$$

If we multiply the first inequality above by $(u+w)'$, and the equation by v', then add the results side by side and make the possible reductions of similar terms by utilizing among others the antisymmetry of f and g, we conclude that

$$\bigcup_{\substack{u,v,w\\ u\geqslant 0\\ w\geqslant 0\\ r>0}} \begin{pmatrix} (u+w)'w \leqslant 0 \\ r'w > 0 \end{pmatrix}.$$

This is, however, a contradiction. Theorem 2 is thus proved.

COROLLARY 2. *We have*

$$\bigcap_{\substack{e,f,g\\ -f=f'\\ -g=g'}} \bigcap_{\substack{x,y\\ x\geqslant 0}} \begin{pmatrix} fx-ey \geqslant 0 \\ fx-ey+x > 0 \\ e'x-gy = 0 \end{pmatrix} \subset \begin{pmatrix} x'x > 0 \\ x \mid (fx-ey) > 0 \\ x'ey = 0 \end{pmatrix}.$$

By $x \mid y > 0$ we understand here and in the sequel the conjunction of all disjunctions of the form $x_i > 0 \mid y_i > 0$, where y_i and x_i are components of vectors y and x that correspond to each other.

Proof. To see this let us consider the first three relations. By multiplying the third of them by y' we get $y'ex = 0$ because of the antisymmetry of g, and $x'ey = 0$ by transposition. If we multiply the second one by x' and use the antisymmetry of f as well as the relation $x'ey = 0$, we obtain $x'x > 0$. Next we obtain $x'(fx-ey) = 0$ by multiplying the first relation by x' and using the antisymmetry of f and the relation $x'ey = 0$. This together with $fx-ey \geqslant 0$, $x \geqslant 0$ and $fx-ey+x > 0$ implies $x \mid (fx-ey) > 0$.

COROLLARY 3. *We have*

$$\bigcap_{\substack{e,f,g\\ -f=f'\\ -g=g'}} \bigcup_{\substack{x,y\\ x\geqslant 0}} \begin{pmatrix} fx-ey \geqslant 0 \\ fx-ey+x > 0 \\ e'x-gy = 0 \end{pmatrix} \cap \begin{pmatrix} x'x > 0 \\ x \mid (fx-ey) > 0 \\ x'ey = 0 \end{pmatrix}.$$

Proof. The first three relations are just in Theorem 2, and the last three may be added in view of Corollary 2.

COROLLARY 3a. *We have*

$$\bigcap_{\substack{e,f,g\\ -f=f'\\ -g=g'}} \bigcup_{\substack{x,y\\ x\geqslant 0\\ x'>0}} \begin{pmatrix} fx-ey \geqslant 0 \\ x \mid (fx-ey) > 0 \\ e'x-gy = 0 \end{pmatrix}.$$

This corollary is clearly a formal consequence of Corollary 3. Both are, however, equivalent.

COROLLARY 4. *We have*

$$\bigcap_{-f=f'} \bigcup_{\substack{x\geqslant 0\\ x'>0}} \begin{pmatrix} fx \geqslant 0 \\ fx+x > 0 \\ x \mid fx > 0 \end{pmatrix}.$$

Proof. Put $e = g = 0$ in Corollary 3.

This corollary is equivalent to Theorem 7 in [4], p. 16, and formally stronger than Theorem 5 in [4], p. 13.

Corollary 5. *We have*

$$\bigcap_{a,b,c,d} \bigcup_{\substack{u,v,p,q\\ u\geqslant 0\\ v\geqslant 0}} \begin{pmatrix} a'v+b'q \geqslant 0 \\ a'v+b'q+u > 0 \\ c'v+d'q = 0 \\ -au-cp \geqslant 0 \\ -au-cp+v > 0 \\ -bu-dp = 0 \\ u'u+v'v > 0 \\ u \mid (a'v+b'q) > 0 \\ v \mid (-au-cp) > 0 \\ q'bu+v'cp = 0 \end{pmatrix}.$$

Proof. To see this substitute

$$f = \begin{pmatrix} 0 & a' \\ -a & 0 \end{pmatrix}, \quad -e = \begin{pmatrix} 0 & b' \\ -c & 0 \end{pmatrix}, \quad -g = \begin{pmatrix} 0 & d' \\ -d & 0 \end{pmatrix},$$

$$x = \begin{pmatrix} a \\ v \end{pmatrix}, \quad y = \begin{pmatrix} p \\ q \end{pmatrix}$$

in Corollary 3.

Corollary 5 is only formally stronger than Theorem 4 in [4], p. 12, and it is equivalent to Theorem 6 in [4], p. 14.

Corollary 5a. *We have*

$$\bigcap_{a,b,c,d} \bigcup_{\substack{u,v,p,q\\ u\geqslant 0\\ v\geqslant 0\\ \binom{u}{v}' > 0}} \begin{pmatrix} a'v+b'q \geqslant 0 \\ c'v+d'q = 0 \\ -au-cp \geqslant 0 \\ -bu-dp = 0 \\ u \mid (a'v+b'q) > 0 \\ v \mid (-au-cp) > 0 \end{pmatrix}.$$

Proof. Put in Corollary 3a the substitutions that were used in the proof of Corollary 5.

Corollary 6 (formally stronger than Theorem 1 in [4], p. 8). *We have*

$$\bigcap_{b} \bigcup_{\substack{u,q\\ u\geqslant 0}} \begin{pmatrix} b'q \geqslant 0 \\ b'q+u > 0 \\ bu = 0 \\ u \mid b'q > 0 \end{pmatrix}.$$

Proof. Put $a = c = d = 0$ in Corollary 5.

COROLLARY 7 (formally stronger than Theorem 2 in [4], p. 9). *We have*

$$\bigcap_{b,d} \bigcup_{\substack{p,q,u \\ u\geqslant 0}} \begin{pmatrix} b'q \geqslant 0 \\ b'q+u > 0 \\ d'q = 0 \\ bu+dp = 0 \\ q'bu = 0 \\ u \mid b'q > 0 \end{pmatrix}.$$

Proof. Put $a = c = 0$ in Corollary 5.

COROLLARY 8 (formally stronger than Theorem 3 in [4], p. 11). *We have*

$$\bigcap_{a} \bigcup_{\substack{u\geqslant 0 \\ v\geqslant 0}} \begin{pmatrix} a'v \geqslant 0 \\ a'v+u > 0 \\ -au \geqslant 0 \\ -au+v > 0 \\ u \mid a'v > 0 \\ v \mid -au > 0 \end{pmatrix}.$$

Proof. Put $b = c = d = 0$ in Corollary 5.

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