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IMAGES OF COMPACT 0-*DIMENSIONAL SEMIGROUPS*

BY

AUGUST LAU (DENTON, TEXAS)

Every groupoid considered in the paper is equipped with a Hausdorff topology which makes the operation jointly continuous. A *semilattice* is a commutative semigroup in which each element is idempotent; it is a *Lawson semilattice* if it has a basis of neighborhoods each of which is a subsemigroup. A compact semigroup S is a *Z-image* if there exists a continuous homomorphism from a compact 0-dimensional semigroup onto S. The first hint of using Z-images is the following theorem:

THEOREM 1. *Let S be a compact semilattice. Then the following statements are equivalent*:

(1) *S is a Lawson semilattice.*

(2) *S is a continuous homomorphic image of a compact* 0-*dimensional semilattice.*

(3) *S is a continuous homomorphic image of a compact* 0-*dimensional semigroup.*

It was well known that (1) and (2) are equivalent (see [3] and [6]). A compact groupoid S is *finitely neighborable* (f.n.) if, given any open cover $\mathscr{U}$ of S, there exists a finite refinement $\mathscr{V}$ such that

$$S = \bigcup \{\operatorname{int} V \mid V \in \mathscr{V}\},$$

and if $A, B \in \mathscr{V}$, then there exists $C \in \mathscr{V}$ such that $AB \subseteq C$. It turns out that compact 0-dimensional semigroups and their continuous homomorphic images are f.n. (see [4]). So a compact semilattice which is a Z-image is f.n. For a compact semilattice, f.n. is equivalent to being Lawson [4].

Theorem 1 shows the importance of Z-images. One would like to obtain an intrinsic characterization of semigroups which are Z-images. However, for groupoids, there is a solution [5]:

THEOREM 2. *Let S be a compact groupoid. Then the following statements are equivalent*:

(1) *S is a continuous homomorphic image of a compact subgroupoid of a product of finite groupoids.*

(2) *S is f.n.*

Unfortunately, the author cannot adapt the groupoid proof to the semigroup situation. So it remains:

PROBLEM 1 (**P 1049**). Let S be a compact semigroup. Is it true that S is a Z-image iff S is f.n.?

Hofmann in [1] used the concept of *ultrametric*, i.e.,

$$d(ab, cd) \leqslant \max\big(d(a, c), d(b, d)\big).$$

It turns out that f.n. is stronger than ultrametric.

LEMMA 1. *If S is a compact metric f.n. semigroup, then S is ultrametrizable.*

Proof. Let $N_1, N_2, \ldots$ be a countable base for its uniformity. Since S is compact, for each n there exists a finite open cover $\mathscr{U}_n$ such that

$$\bigcup\{U \times U \mid U \in \mathscr{U}_n\} \subseteq N_n.$$

Let $\mathscr{V}_n$ be the refinement of $\mathscr{U}_n$ given by the definition of f.n. If $U, V \in \mathscr{V}_n$, then

$$(U \times U)(V \times V) \subseteq W \times W \quad \text{for some } W \in \mathscr{V}_n.$$

Thus $\bigcup\{V \times V \mid V \in \mathscr{V}_n\}$ is a subsemigroup of $S \times S$. This is sufficient to give S an ultrametric (see [1], p. 283).

LEMMA 2. *Let S be a compact semigroup. Then a Z-image implies f.n. which implies an inverse limit of ultrametrizable compact semigroups.*

Proof. Z-images are f.n. [4]. Let S be f.n. Then S is an inverse limit of compact metric semigroups (see [2] or [7]). Each factor of the limit is f.n., since S is f.n. By Lemma 1, each factor is ultrametrizable.

Before proceeding to the equivalence of the three concepts for commutative semigroups, some lemmas are needed.

LEMMA 3. *If S is a compact ultrametrizable semigroup, then each maximal subgroup is* 0*-dimensional.*

For the proof, see [1], p. 282.

Notation. If F is a subset of a semigroup, then F^* denotes the smallest closed semigroup generated by F.

LEMMA 4. *If S is a compact commutative semigroup where each maximal subgroup is* 0*-dimensional and F is a finite subset of S, then F^* is* 0*-dimensional.*

Proof. If G is a compact subgroup of S and $x \in S$, then xG is 0-dimensional, since xG is homeomorphic to G/H, where $H = \{g \in G \mid xg = xe\}$ (e is the identity of G). If $y \in F$, then $\{y\}^*$ is 0-dimensional since its minimal ideal is a compact group. Note that S is commutative. Hence F^* is the countable union of finite products of singletons or finite products of points with compact 0-dimensional groups. A finite product of compact

0-dimensional groups is the quotient of the Cartesian product of the groups, and hence is 0-dimensional. By the sum theorem in dimension theory, F^* is 0-dimensional.

If S is a compact ultrametric semigroup, then 2^S is the hyperspace of compact non-empty subsets of S with the Hausdorff metric which is an ultrametric on 2^S (2^S is a semigroup under set product). It is equivalent to the Vietoris topology generated by

$$\langle U_1, U_2, \ldots, U_n\rangle = \Big\{A \in 2^S \,|\, A \subseteq \bigcup_{i=1}^{n} U_i \text{ and } A\cap U_i \neq \emptyset \text{ for each } i\Big\},$$

where each U_i is open in S.

THEOREM 3. *If S is a compact commutative semigroup, then the following statements are equivalent*:

(1) *S is a Z-image.*

(2) *S is f.n.*

(3) *S is an inverse limit of ultrametrizable compact semigroups.*

Proof. By Lemma 2, (1)$\Rightarrow$(2)$\Rightarrow$(3). It is sufficient to prove that if S is a compact commutative ultrametric semigroup, then S is a Z-image. Construct a sequence of finite closed covers of S, denoted by $\mathscr{C}_1, \mathscr{C}_2, \ldots,$ such that

(a) $\operatorname{mesh}\mathscr{C}_n \leqslant 1/n$,

(b) if $A \in \mathscr{C}_n$, then A is the union of some elements of $\mathscr{C}_{n+1}$.

Note that if $\mathscr{C}$ is a finite closed cover of S and $\operatorname{mesh}\mathscr{C} \leqslant t$, then $\operatorname{diam} A_1A_2\ldots A_n \leqslant t$ for $A_1, A_2, \ldots, A_n \in \mathscr{C}$, since S has an ultrametric. Hence $\operatorname{mesh}\mathscr{C}^* \leqslant t$, where $\mathscr{C}^*$ is the compact semigroup generated by $\mathscr{C}$ in 2^S. Also $\mathscr{C}^*$ is 0-dimensional by Lemmas 3 and 4.

Let

$$G = \Big\{(A_n) \in \prod_{n=1}^{\infty} \mathscr{C}_n^* \,|\, A_n \supseteq A_{n+1} \text{ for all } n = 1, 2, \ldots\Big\}.$$

Then G is a subsemigroup of the Cartesian product. To show that G is closed, let $A_{n+1} \nsubseteq A_n$ for some n. Choose open sets U and V so that $A_n \subseteq U$ and $A_{n+1}\cap V \neq \emptyset$ and $U\cap V = \emptyset$. Let

$$W = \underbrace{2^S \times \ldots \times 2^S}_{n-1 \text{ factors}} \times \langle U\rangle \times \langle S, V\rangle \times 2^S \times \ldots$$

Then $W\cap G = \emptyset$.

Define $f\colon G \to S$ by $f\big((A_n)\big) = \bigcap A_n$ which is a point since $\operatorname{mesh}\mathscr{C}_n^* \to 0$. To show continuity, let $\bigcap A_n^* \in U$ which is open in S. Then $A_N \subseteq U$ for some N. Let

$$W = \underbrace{2^S \times \ldots \times 2^S}_{N-1 \text{ terms}} \times \langle U\rangle \times 2^S \times \ldots$$

Then $f(W \cap G) \subseteq U$.

To show that f is a homomorphism, let $(A_n), (B_n) \in G$. Then

$$\left(\bigcap A_n\right)\left(\bigcap B_n\right) \subseteq \bigcap A_n B_n.$$

But $\operatorname{diam} A_n B_n \to 0$ implies

$$\left(\bigcap A_n\right)\left(\bigcap B_n\right) = \bigcap A_n B_n.$$

To show "onto", let $x \in S$. Then $x \in A_1 \in \mathscr{C}_1$. Since A_1 is a union of some elements in $\mathscr{C}_2$, $x \in A_2 \subseteq A_1$ for some $A_2 \in \mathscr{C}_2$. One can choose a sequence $A_1, A_2, \ldots$ in G so that $x = \bigcap A_n$.

PROBLEM 2 (**P 1050**). If S is a compact ultrametric semigroup and F is a finite subset, is F^* 0-dimensional?

If Problem 2 has a positive solution, then Problem 1 has a positive solution (the same proof as for Theorem 3).

COROLLARY. *The interval* $[0, 1/2]$ *under real multiplication is a Z-image.*

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DEPARTMENT OF MATHEMATICS
NORTH TEXAS STATE UNIVERSITY
DENTON, TEXAS

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