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LOCALLY CONVEX SPACES WITH FACTORIZATION PROPERTY

BY

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0. Introduction. In this paper* we present three new classes of locally convex spaces: spaces with factorization property, pseudo-barrelled spaces, and spaces with s-factorization property. These spaces arise in the study of dilations of operator valued functions in non-Banach spaces. The aim of the paper is to describe a place of these spaces in the large class of locally convex spaces.

In Section 1 we define the class of locally convex spaces with factorization property. This class is large and contains pseudo-barrelled spaces and the spaces with s-factorization property. We also give (cf. (1.4)) an example of a locally convex space which does not have the factorization property. In Section 2 we study the class of pseudo-barrelled spaces. This class contains plenty of well-known spaces: quasi-barrelled spaces, and – in particular – barrelled and bornological spaces, $\mathcal{GF}$-spaces, spaces with mixed topologies, and some generalized inductive-limits. In Section 3 we present the spaces with s-factorization property. This class contains every barrelled space. Therefore, every barrelled space is pseudo-barrelled and has the s-factorization property. We give examples of locally convex spaces with factorization property: 1° spaces which are pseudo-barrelled and do not have the s-factorization property (cf. (3.4)) and 2° spaces which have the s-factorization property and are not pseudo-barrelled (cf. (2.3)).

Applications of the spaces with factorization property to the theory of positive definite kernels (cf. [3] and [12] for the classical results and [14], [19], [16] for the Banach space case) and to the dilation theory (cf. [17], [15] for the classical results and [8], [20] for the Banach space case) are of some interest – for details see [9] and [6], [7].

1. Spaces with factorization property. Let E be a complex Hausdorff locally convex space and E' its topological dual.

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* The results of the paper are taken from the author's Ph. D. Thesis, Wroclaw Technical University, 1977.

By $\bar{L}(E, E')$ $(C\bar{L}(E, E'))$ we denote the space of all antilinear (continuous antilinear) mappings from E into E' being equipped with the strong topology $\beta(E', E)$, i.e., the topology of uniform convergence on the bounded subsets of E.

$CL(E, H)$ denotes the space of all continuous linear mappings from E into a Hilbert space H.

For $A \in CL(E, H)$, we define the adjoint mapping $A' \in C\bar{L}(H, E')$ by the formula

$$(1.1) \qquad (A'f)(x) = (Ax, f), \quad f \in H, \ x \in E,$$

where $(\cdot, \cdot)$ denotes the inner product in H.

We say that the mapping $A \in \bar{L}(E, E')$ is *positive* if $(Ax)(x) \geqslant 0$ for each $x \in E$.

Now, we introduce a new class of locally convex spaces.

(1.2) Definition. A locally convex space E has the *factorization property* if for each inner product $(\cdot, \cdot)$ defined on E and satisfying the condition

(1.3) $p_B(x) = \sup_{y \in B} |(x, y)| < \infty$ for each bounded subset $B \subset E$ and the semi-norm $p_B(x)$ is continuous,

the function $E \ni x \to (x, x)$ is continuous.

Next, we give an example of a locally convex space which does not have the factorization property.

(1.4) Example. Let s_0 be the vector space of all complex sequences having only a finite number of terms different from zero. We introduce a topology τ in s_0 by the family of semi-norms

$$(1.5) \qquad p_{N_0, \{M_n\}}(x) = \sum_{n \in N_0} M_n |x_n|, \qquad x = (x_1, x_2, \ldots) \in s_0,$$

where N_0 is an arbitrary subset of positive integers with density equal to zero, i.e.,

$$\lim_{n \to \infty} \frac{\text{card}\,\{N_0 \cap \{1, 2, \ldots, n\}\}}{n} = 0,$$

and $\{M_n\}$ is an arbitrary sequence of non-negative numbers.

Note that a subset $B \subset s_0$ is τ-bounded iff there exist a natural number n_0 and a constant $C > 0$ such that the condition $x = (x_n)_{n=1}^{\infty} \in B$ implies $x_n = 0$ for $n > n_0$ and $|x_n| < C$.

We define an inner product $(\cdot, \cdot)$ in a locally convex space (s_0, τ) by

$$(1.6) \qquad (x, y) = \sum_{k=1}^{\infty} x_k \bar{y}_k, \qquad x, y \in s_0.$$

Observe that the inner product (1.6) satisfies (1.3), but the function

$$s_0 \ni x \to (x, x) = \sum_{k=1}^{\infty} |x_k|^2$$

is not continuous because for each constant $M > 0$ and for each semi-norm $p(x)$ in the family (1.5) there exists $x \in s_0$ such that $(x, x) > Mp(x)$.

(1.7) THEOREM. *Let E be a complex locally convex space and E' its topological dual with the strong topology. Then the following conditions are equivalent:*

(i) *E has the factorization property.*

(ii) *For each positive mapping $R \in C\bar{L}(E, E')$ the function $E \ni x \to (Rx)(x)$ is continuous.*

(iii) *For each positive mapping $R \in C\bar{L}(E, E')$ there exist a Hilbert space H and a square root $T \in CL(E, H)$ of R, i.e., $R = T'T$. Moreover, if H is minimal in the sense that it is generated by the elements of the form Tx, $x \in E$, then H and T are unique up to unitary equivalence.*

Proof. (i)$\Rightarrow$(ii). Let E have the factorization property. Define an inner product in E by the formula $(y, x) = (Rx, y)$, $x, y \in E$, where $R \in C\bar{L}(E, E')$ is positive. This inner product satisfies (1.3). Indeed, if $B \subset E$ is a bounded subset, then $p_B(x) = \sup_{y \in B} |(Rx)(y)|$ is finite for each $x \in E$ because Rx is bounded on B. Moreover, $p_B(x)$ is continuous, which follows from the assumption that $R \in C\bar{L}(E, E')$. Consequently, by (i), the function $E \ni x \to (x, x) = (Rx)(x)$ is continuous.

(ii)$\Rightarrow$(iii). Let $R \in C\bar{L}(E, E')$ be positive. By (2.7) in [9] there exist a Hilbert space H and a linear mapping $T: E \to H$ which is continuous on the bounded subsets of E and such that $R = T'T$. Hence $\|Tx\|^2 = (Rx)(x)$ and, by (ii), T is continuous, i.e., $T \in CL(E, H)$.

(iii)$\Rightarrow$(i). Suppose, conversely, that $(\cdot, \cdot)$ is an inner product in E for which (1.3) holds but the function $E \ni x \to (x, x)$ is not continuous. Then the mapping $R: E \to E'$ given by the formula $(Rx)(y) = (y, x)$, $x, y \in E$, is well defined, positive, and $R \in C\bar{L}(E, E')$, but the function $E \ni x \to (Rx)(x)$ is not continuous. On the other hand, it follows from (iii) that $(Rx)(x) = \|Tx\|^2$ for each $x \in E$ and, consequently, the function $E \ni x \to (Rx)(x)$ is continuous because $T \in CL(E, H)$. This contradiction completes the proof.

2. Pseudo-barrelled spaces. In this section we introduce and discuss a new class of locally convex spaces — pseudo-barrelled spaces. Pseudo-barrelled spaces have the factorization property (Theorem (2.2)) and contain plenty of well-known spaces: quasi-barrelled spaces, and — in particular — barrelled and bornological spaces, $\mathcal{LF}$-spaces, spaces with mixed topologies in the Wiweger and Persson sense, and some generalized inductive-limits.

We also give an example (Example (2.3)) of a locally convex space with factorization property, which is not a pseudo-barrelled space.

(2.1) Definition. We say that a locally convex space E is *pseudo-barrelled* if each semi-norm $p(x)$ on E, being lower semi-continuous and continuous on the bounded subsets of E, is continuous.

(2.2) THEOREM ([9], Proposition (3.1)). *Each pseudo-barrelled space has the factorization property.*

The following example shows that the converse of Theorem (2.2) is not true.

(2.3) Example. Let Z be an uncountable set and $s_0(Z)$ the vector space of all complex functions $x: Z \to C$ having only a finite number of values different from zero.

We introduce a topology τ in $s_0(Z)$ by all inner products

$$(x, y) = \sum_{u,v\in Z} a_{u,v} x_u \bar{y}_v, \qquad x = (x_u)_{u\in Z}, \ y = (y_u)_{u\in Z} \in s_0(Z), \tag{2.4}$$

i.e., by all positive definite forms on $s_0(Z)$.

It follows from Definition (1.2) that a locally convex space $(s_0(Z), \tau)$ has the factorization property.

Now, we shall prove that the space $(s_0(Z), \tau)$ is not pseudo-barrelled.

Define a semi-norm $p(x)$ on $s_0(Z)$ by

$$p(x) = \sum_{u\in Z} |x_u|, \qquad x = (x_u)_{u\in Z} \in s_0(Z).$$

The semi-norm $p(x)$ is lower semi-continuous and continuous on the τ-bounded subsets of $s_0(Z)$ (each τ-bounded set is finite dimensional and bounded in an ordinary sense) but is not continuous. Indeed, suppose, conversely, that $p(x)$ is continuous. Then there exists a positive definite matrix $(a_{u,v})$ such that

$$\sum_{u\in Z} |x_u| \leqslant \Big[\sum_{u,v\in Z} a_{u,v} x_u \bar{x}_v\Big]^{1/2}, \qquad x = (x_u)_{u\in Z} \in s_0(Z). \tag{2.5}$$

Inequality (2.5) shows that the identity map I on the inner product space $s_0(Z)$ (with a strictly positive inner product given by the matrix $(a_{u,v}+\delta_{u,v})$ and formula (2.4)) into the normed space $l^1(Z)$ (i.e., a set $s_0(Z)$ with the norm $\|x\| = \sum_{u\in Z} |x_u|$) is continuous.

Denote by $\tilde{s}_0(Z)$ the completion of $s_0(Z)$ and by $A: \tilde{s}_0(Z) \to l^1(Z)$ the linear continuous extension of I. The space $\tilde{s}_0(Z)$ is reflexive and $A: \tilde{s}_0(Z) \to l^1(Z)$ is compact. Hence the set $A(\tilde{s}_0(Z))$ is separable. On the other hand, the set $A(\tilde{s}_0(Z))$ contains an uncountable subset of functions δ_u, $u \in Z$ ($\delta_u(x) = 1$ for $x = u$ and $\delta_u(x) = 0$ for $x \neq u$). Since the distance $\varrho(\delta_u, \delta_v)$ equals 2 for $u \neq v$, $A(\tilde{s}_0(Z))$ is not separable. This contradiction completes the proof that $(s_0(Z), \tau)$ is not pseudo-barrelled.

Now, we answer the question which spaces contain the class of pseudo-barrelled spaces.

Let E be a locally convex space.

Every closed, absorbent, absolutely convex subset of E is called a *barrel.*

(2.6) Following Bourbaki [4] we say that a locally convex space E is *barrelled* if each barrel in E is a neighbourhood of 0 or, equivalently, if each lower semi-continuous semi-norm $p(x)$ on E is continuous.

Every locally convex Baire space and, in particular, every Fréchet space (metrizable and complete locally convex space) is barrelled.

(2.7) A locally convex space E is said to be *bornological* if every absolutely convex set in E which absorbs all bounded subsets of E is a neighbourhood of 0 or, equivalently, if each semi-norm $p(x)$ on E, being bounded on the bounded subsets of E, is continuous.

All metrizable locally convex spaces, as well as LB and LF spaces, are bornological.

(2.8) A locally convex space E is called *quasi-barrelled* if every barrel in E which absorbs all bounded sets of E is a neighbourhood of 0 or, equivalently, if each semi-norm $p(x)$ on E, being lower semi-continuous and bounded on the bounded sets of E, is continuous.

Clearly, every barrelled space as well as every bornological space is quasi-barrelled.

(2.9) COROLLARY. *Every quasi-barrelled space (in particular, every barrelled and every bornological space) is pseudo-barrelled. Consequently* (by Theorem (2.2)), *it has the factorization property.*

The corollary follows immediately from Definitions (2.8) and (2.1).

Grothendieck introduced in [10] the class of $\mathcal{DF}$-spaces. A locally convex space E is called a *$\mathcal{DF}$-space* if it has a fundamental sequence of bounded sets and if each barrel in E, which absorbs every set in E and which is an intersection of at most countably many closed, absolutely convex neighbourhoods of 0, is a neighbourhood of 0.

(2.10) Remark. We observe that the above definition of $\mathcal{DF}$-space is equivalent to the following: a space E is a $\mathcal{DF}$-space iff it has a fundamental sequence of bounded sets and if each semi-norm $p(x)$ on E, being lower semi-continuous and bounded on the bounded subsets of E and being the least upper bound of a collection of at most countably many continuous semi-norms on E, is continuous.

(2.11) PROPOSITION. *Every $\mathcal{DF}$-space E is pseudo-barrelled.*

Proof. Denote by μ the topology in E. We remark that μ is the finest locally convex topology on E, which is identical with itself on the μ-

bounded subsets of E. In fact, if μ' is the finest locally convex topology on E which is identical with μ on the μ-bounded subsets of E, then by Grothendieck's theorem (cf. [13], § 29.3.7) the identity map $I: (E, \mu) \to (E, \mu')$ is continuous and $\mu = \mu'$. Hence each semi-norm $p(x)$ on E, being lower semi-continuous and continuous on the bounded subsets of E, is continuous.

By a *two-norm space* we mean a triplet $(E, \|\cdot\|, \|\cdot\|^*)$ of a vector space E and two norms $\|\cdot\|$, $\|\cdot\|^*$, the second being dominated by the first. A sequence $(x_n)_{n=1}^{\infty}$ is said to be *convergent* to x_0 in the two-norm sense (or γ-*convergent*) if

$$\sup_n \|x_n\| < \infty \quad \text{and} \quad \|x_n - x_0\|^* \to 0.$$

The theory of the two-norm spaces has been developed by Alexiewicz and Semadeni [1], [2].

Wiweger introduced in [21], [22] a topology $\tilde{\tau}$ (called the *mixed topology*) in a two-norm space $(E, \|\cdot\|, \|\cdot\|^*)$ such that the sequence $(x_n)_{n=1}^{\infty}$ is γ-convergent to x_0 iff $x_n \to x_0$ in the $\tilde{\tau}$-topology ([21], (2.3)).

Suppose that there are two locally convex topologies τ and τ^* in a vector space E. Let $\mathcal{U}(\tau)$ and $\mathcal{U}(\tau^*)$ be bases of convex neighbourhoods of 0 in topologies τ and τ^*, respectively. For each sequence $U_n^* \in \mathcal{U}(\tau^*)$ and for each $U \in \mathcal{U}(\tau)$ Wiweger [21], [22] defines the set U^γ as

$$(2.12) \qquad U^\gamma = \bigcup_{n=1}^{\infty} (U_1^* \cap U + U_2^* \cap 2U + \ldots + U_n^* \cap nU)$$

and proves the following:

(2.13) The family of all sets of the form (2.12) is a basis of neighbourhoods of 0 in the locally convex topology τ^γ, called the *mixed topology* on E.

(2.14) ([22], 2.1.1) The mixed topology τ^γ satisfies the inequality $\tau^* \leqslant \tau^\gamma$. Moreover, if $\tau^* \leqslant \tau$, then $\tau^\gamma \leqslant \tau$.

(2.15) ([22], 2.2.1) The mixed topology τ^γ coincides with τ^* on the τ-bounded subsets of E.

(2.16) ([22], 2.2.5) If τ is a norm topology, then τ^γ is the finest locally convex topology on E which is identical with τ^* on the τ-bounded subsets of E.

Wiweger [22] gives several examples of spaces with mixed topologies as well as spaces with mixed topologies which are neither barrelled nor bornological.

Persson [18] generalized the concept of two-norm space introducing bitopological spaces.

(2.17) ([18]). A triplet (E, τ, τ^*) of a vector space E and of two topologies τ, τ^* on E such that every τ-bounded subset of E is τ^*-bounded is called a *bitopological space*.

The finest locally convex topology on E which is identical with τ^* on the τ-bounded subsets of E is called the *mixed topology* on E and is denoted by τ^γ.

(2.18) Remark. If τ is a norm topology such that every τ-bounded subset on E is τ^*-bounded, then the mixed topology in the Wiweger sense is the mixed topology in the Persson sense.

This follows directly from the above definition and from (2.16).

(2.19) PROPOSITION. *A locally convex space E with the mixed topology τ^γ in the Persson sense* (2.17) *determined by the topologies τ and τ^* (and every mixed topology determined by τ and τ^* as in* (2.18)) *is pseudo-barrelled. Consequently* (by Theorem (2.2)), *τ^γ has the factorization property.*

Proof. Let (E, τ, τ^*) be a bitopological space, τ^γ the mixed topology, and assume $p(x)$ to be a lower semi-continuous semi-norm on E which is continuous on the τ^γ-bounded subsets of E.

We have to prove that the semi-norm $p(x)$ is continuous on (E, τ^γ).

Denote by $(S_\alpha)_{\alpha\in A}$ the family of all τ-bounded subsets of E. The restriction of $p(x)$ to S_α (denoted by $p|S_\alpha$) is continuous in the topology $\tau^*|S_\alpha$ induced by τ^* on S_α. Indeed, by Proposition (1.1) in [18], every τ-bounded subset S_α $(\alpha\in A)$ of E is τ^γ-bounded and $p|S_\alpha$ is $(\tau^\gamma|S_\alpha)$-continuous. This implies the $(\tau^*|S_\alpha)$-continuity. Finally, for every τ-bounded set S_α the restriction of $p(x)$ to S_α is τ^*-continuous and, by (2.17), the semi-norm $p(x)$ is τ^γ-continuous.

Garling introduced in [5] a generalized inductive-limit topology on a vector space E.

Let a family $(E_\alpha)_{\alpha\in A}$ of vector spaces be given. Each E_α is given a locally convex topology τ_α and for each E_α a linear mapping $i_\alpha\colon E_\alpha \to E$ is defined. For each $\alpha\in A$, a subset S_α of E_α is given. Denote by j_α the restriction of i_α to S_α.

(2.20) ([5]) The *generalized inductive-limit topology* on E, induced by the family $\{(E_\alpha, \tau_\alpha, i_\alpha, S_\alpha)\}_{\alpha\in A}$, is defined to be the finest locally convex topology on E for which each of the mappings $j_\alpha\colon S_\alpha \to E$ is continuous.

(2.21) Remark. It is easy to verify that the topology of every bornological and every $\mathscr{DF}$-space is the mixed topology in the Persson sense and that every mixed topology is the generalized inductive-limit topology.

Now, we show that some generalized inductive-limit topologies (in particular, studied by Garling in [5]) are pseudo-barrelled.

(2.22) PROPOSITION. *The generalized inductive-limit topology on E induced by the family $\{(E_\alpha, \tau_\alpha, i_\alpha, S_\alpha)\}_{\alpha\in A}$, where for each $\alpha\in A$ the set $S_\alpha \subset E'_\alpha$ is bounded and such that*

$$(*) \qquad x\in S_\alpha \Rightarrow ax\in S_\alpha \quad \textit{for } 0\leqslant a\leqslant 1$$

(*e.g.*, *if* S_α *is an absolutely convex bounded set*), *is pseudo-barrelled. Consequently* (by Theorem (2.2)), *it has the factorization property.*

Proof. Let $p(x)$ be a semi-norm on E which is lower semi-continuous and continuous on the bounded subsets of E.

By Corollary 2 in [5], a semi-norm $p(x)$ on E is continuous in the generalized inductive-limit topology if, for each index $\alpha \in A$, the restriction of the semi-norm pj_α to S_α, i.e., $pj_\alpha: S_\alpha \to E$, is continuous.

Since S_α is bounded and satisfies condition (*) and since $j_\alpha: S_\alpha \to E$ is continuous, we deduce that for each $\alpha \in A$ the set $j_\alpha(S_\alpha)$ is bounded in E (cf. [13], § 15.6.3). This, along with the assumption that the semi-norm $p(x)$ is continuous on the set $j_\alpha(S_\alpha)$, yields the continuity of $pj_\alpha: S_\alpha \to E$ and completes the proof.

Finally, we note that the pseudo-barrelled spaces have the following properties:

(2.23) A direct sum and the inductive-limit (cf. [13] for definitions) of pseudo-barrelled spaces are pseudo-barrelled.

(2.24) A closed linear subspace of a pseudo-barrelled space need not be pseudo-barrelled.

3. Spaces with *s*-factorization property. In the preceding two sections we considered locally convex spaces with factorization property.

In the study of dilations of operator valued functions in vector spaces [6]-[9] we define a new subclass of the class of spaces with factorization property.

(3.1) Definition. A locally convex space E has the *s-factorization property* if for each inner product $(\cdot, \cdot)$ defined on E, which is coordinatewise continuous, the function $E \ni x \to (x, x)$ is continuous.

(3.2) Remark. From (3.1) and (1.2) it follows immediately that every space with s-factorization property has the factorization property.

The next theorem is analogous to Theorem (1.7). Its proof is similar to that of (1.7) and will be omitted.

(3.3) THEOREM. *Let E be a complex locally convex space and E' its topological dual. The following conditions are equivalent*:

(i) *E has the s-factorization property.*

(ii) *For each positive mapping $R \in \bar{L}(E, E')$ the function $E \ni x \to (Rx)(x)$ is continuous.*

(iii) *For each positive mapping $R \in \bar{L}(E, E')$ there exist a Hilbert space H and a continuous square root $T \in CL(E, H)$ of R, i.e., $R = T'T$. Moreover, if H is minimal* (cf. (iii) in (1.7)), *then H and T are unique up to unitary equivalence.*

From Theorem (2.2) and (3.2) we know that the classes of pseudo-barrelled spaces and spaces with s-factorization property are both contained in the class of spaces with factorization property.

The space $(s_0(Z), \tau)$ in (2.3) is not pseudo-barrelled. Obviously, it has the s-factorization property.

Now, we give an example of a pseudo-barrelled space which does not have the s-factorization property.

(3.4) Example. Let E be a complex non-separable Hilbert space with the inner product $(\cdot, \cdot)$. By τ and τ^w we denote the norm topology and the weak one on E, respectively.

Define on E a locally convex topology τ_1 by the family of all semi-norms

$$p(x) = \|Px\|, \quad x \in E, \tag{3.5}$$

where P is an orthogonal projection onto a separable linear subspace of (E, τ).

First, we prove that the space (E, τ_1) does not have the s-factorization property.

It is clear that $\tau^w \leqslant \tau_1 \leqslant \tau$ and that $\tau_1 \neq \tau$. Define a mapping $R: (E, \tau_1) \to (E', \beta(E', E))$ by

$$(Rx)(y) = (y, x), \quad x, y \in E. \tag{3.6}$$

The mapping R is well defined, antilinear, and positive. Observe that the topology on E' of uniform convergence on the bounded subsets of (E, τ_1) (the strong topology $\beta(E', E)$) coincides with the norm topology τ on E, since every τ^w-bounded and, in particular, every τ_1-bounded set is τ-bounded (cf. [13], § 20.11.7). Since $\tau_1 \neq \tau$, a mapping $R: (E, \tau_1) \to (E', \beta(E', E))$ defined in (3.6) is not continuous and, by Theorem (3.3), (E, τ_1) does not have the s-factorization property.

Now we prove that the space (E, τ_1) is pseudo-barrelled.

First, we show that (E, τ_1) is σ-barrelled. The Husain definition of σ-barrelled space (cf. [11]) is equivalent to the following: a locally convex space E is σ-barrelled iff each lower semi-continuous semi-norm $p(x)$ on E, which is the least upper bound of at most countably many continuous semi-norms on E, is continuous.

Let $p(x)$ be a lower semi-continuous semi-norm on (E, τ_1) which is the least upper bound of at most countably many continuous semi-norms $p_n(x)$ on (E, τ_1), $n = 1, 2, \ldots$ For each n, there exist an orthogonal projection P_n onto a separable subspace $E_n \subset (E, \tau)$ and a constant $C_n > 0$ such that

$$p_n(x) \leqslant C_n \|P_n x\|, \quad x \in E. \tag{3.7}$$

Denote by P_0 an orthogonal projection onto a separable subspace

$$E_0 = \overline{\bigcup_{n=1}^{\infty} E_n} \subset (E, \tau).$$

By (3.7) we have $p_n(x) \leqslant C_n \|P_0 x\|$, $x \in E$. Hence each semi-norm $p_n(x)$ is equal to 0 for every x in $E_0^\perp$, the orthogonal complement of E_0, and

$$p(x) = 0, \quad x \in E_0^\perp. \tag{3.8}$$

By assumption, the semi-norm $p(x)$ is lower semi-continuous on (E, τ_1) (thus on (E, τ)) and on a barrelled space (E_0, τ). Hence, by (2.6), $p(x)$ is τ-continuous on E_0, i.e., there exists a constant $C > 0$ such that $p(x) \leqslant C\|x\|$, $x \in E_0$. Hence, by (3.8), $p(x) \leqslant C\|P_0 x\|$ for each $x \in E$, i.e., $p(x)$ is τ_1-continuous on E, so the space (E, τ_1) is σ-barrelled.

Now, we observe that the σ-barrelled space (E, τ_1) has a fundamental sequence of bounded sets and, by (2.10), it is a $\mathcal{DF}$-space. Finally, by (2.11), the space (E, τ_1) is pseudo-barrelled.

(3.9) THEOREM ([9], Proposition (3.5)). *Each barrelled space E has the s-factorization property.*

(3.10) Remark. The intersection of the class of all pseudo-barrelled spaces and the class of all spaces with s-factorization property contains the class of all barrelled spaces (cf. (2.9) and (3.9)).

Final remarks. 1. It would be interesting to know what other classes of locally convex spaces have the factorization and s-factorization properties. It turns out also that some of subclasses of m-barrelled and, in particular, σ-barrelled spaces introduced by Husain in [11] have factorization properties.

2. In [6] it was shown that an analogue of B. Sz.-Nagy's dilation theorem holds in a locally convex space E iff E has the factorization property. Moreover, in [9] it was shown that an analogue of the Aronszajn-Kolmogorov kernel theorem holds in a locally convex space E iff E has also the factorization property.

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