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FRÉCHET ALGEBRAS WITH ORTHOGONAL BASIS

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Introduction. Let $(A, \|\cdot\|_k)$ be a Fréchet algebra with unity and with the Schauder basis $(e_n)_{n=1}^{\infty}$ such that $e_n e_m = \delta_{n,m} e_n$ for all $n, m \in N$, where $\delta_{n,m}$ is the Kronecker symbol. Such algebras were investigated for the first time in [3] under the assumption that they are m-convex.

Here we consider such algebras without assumption of m-convexity. We give necessary and sufficient conditions for such an algebra to be isomorphic to the algebra s of all complex sequences with pointwise algebraic operations and the topology of pointwise convergence. We give also sufficient conditions under which all multiplicative linear functionals on such an algebra are continuous.

In the sequel we shall always assume that $(A, \|\cdot\|_k)$ is a Fréchet algebra with unity and an orthogonal basis $(e_n)_{n=1}^{\infty}$, i.e. $e_n e_m = \delta_{n,m} e_n$ for all $n, m \in N$.

An algebra A is called *m-convex* if $\|xy\|_k \leqslant \|x\|_k \|y\|_k$ for all $x, y \in A$ and $k \in N$. In the general case we may assume that $\|xy\|_k \leqslant \|x\|_{k+1} \|y\|_{k+1}$ (see [4]).

We now give some fundamental facts concerning the algebra $(A, \|\cdot\|_k)$.

1. Let

$$x_i = \sum_{n=1}^{\infty} e_n^*(x_i) e_n \in A, \quad i = 1, 2.$$

Then

$$x_1 x_2 = \lim_{n\to\infty} \Big[\Big(\sum_{k=1}^{n} e_k^*(x_1) e_k\Big)\Big(\sum_{k=1}^{n} e_k^*(x_2) e_k\Big)\Big] = \sum_{n=1}^{\infty} e_n^*(x_1) e_n^*(x_2) e_n,$$

i.e.

$$e_n^*(x_1 x_2) = e_n^*(x_1) e_n^*(x_2) \quad (n \in N,\ x_1, x_2 \in A);$$

hence $e_n^* \in \mathfrak{M}(A)$, where $\mathfrak{M}(A)$ denotes the set of all non-zero continuous multiplicative linear functionals on A.

Conversely, every element of $\mathfrak{M}(A)$ is of the form $f(x) = e_n^*(x)$ for some n. Indeed, if $f \in \mathfrak{M}(A)$, then

$$f(e) = f\Big(\sum_{n=1}^{\infty} e_n\Big) = 1$$

and there exists a positive integer n_0 such that $f(e_{n_0}) \neq 0$. Let $x \in A$. Then

$$f(e_{n_0}x) = e^*_{n_0}(x) f(e_{n_0}) \quad \text{and} \quad f(e_{n_0}x) = f(e_{n_0}) f(x).$$

Hence $f(x) = e^*_{n_0}(x)$, i.e. $\mathfrak{M}(A) = \{e^*_1, e^*_2, \ldots\}$. Therefore, if f is a non-trivial discontinuous multiplicative linear functional defined on A, then $f(e) = 1$ and $f(e_n) = 0$ for $n \in N$.

2. Let

$$\begin{aligned} P &= \{(t_n)_{n=1}^{\infty} \in s\colon \sum_{n=1}^{\infty} t_n e_n \in A\} \\ &= \{(t_n)_{n=1}^{\infty} \in s\colon \text{there exists } x \in A \text{ such that } e^*_n(x) = t_n;\ n \in N\}. \end{aligned}$$

The set P is called the *basis field* of $(e_n)_{n=1}^{*}$.

We have the following

THEOREM 0.1. *If* $t_1 \geqslant t_2 \geqslant \ldots \geqslant t_n \geqslant t_{n+1} \geqslant \ldots \geqslant 0$, $t_n \to 0$, *then* $(t_n)_{n=1}^{\infty}$ *is in* P.

Proof. Let $q > p$ and k be positive integers. Then

$$\sum_{n=p}^{q} t_n e_n = e_p(t_p - t_{p+1}) + \ldots + (e_p + \ldots + e_{q-1})(t_{q-1} - t_q) + (e_p + \ldots + e_q) t_q.$$

Hence

$$\Big\| \sum_{n=p}^{q} t_n e_n \Big\|_k \leqslant \big(\sup_{p \leqslant i \leqslant q} \|e_p + \ldots + e_i\|_k \big) t_p.$$

Thus there exists $x_0 \in A$ such that

$$x_0 = \sum_{n=1}^{\infty} t_n e_n.$$

The proof is complete.

The algebra s is an m-convex Fréchet algebra with the orthogonal basis consisting of the vectors $e_n = (0, \ldots, 0, \underset{n}{1}, 0, \ldots)$ for $n \in N$. We shall now give another example of such an algebra. Let $a_{k,n} > 0$, $k, n \in N$. Assume that

(a) $\sum_{n=1}^{\infty} a_{k,n} < \infty$, $k \in N$;

(b) $A_k = \sum_{n=1}^{\infty} a_{k,n}/a^2_{k+1,n} < \infty$, $k \in N$.

Let

$$A = \{x = (t_n)_{n=1}^{\infty}\colon \sum_{n=1}^{\infty} |t_n| a_{k,n} = \|x\|_k < \infty,\ k \in N\}.$$

If $x_i = (t_n^{(i)})_{n=1}^{\infty}$ for $i = 1, 2$, then the formula $x_1 x_2 = (t_n^{(1)} t_n^{(2)})_{n=1}^{\infty}$ defines a multiplication in A. It is evident that

$$\|x_1 x_2\|_k \leqslant A_k \|x_1\|_{k+1} \|x_2\|_{k+1}.$$

Therefore, $(A, \|\cdot\|_k)$ is a Fréchet algebra with the unit element and with the orthogonal basis consisting of the vectors $e_n = (0, \ldots, 0, \underset{n}{1}, 0, \ldots)$, $n \in N$, where $\|e_n\|_k = a_{k,n}$ for $k, n \in N$. From (a) and (b) we have

$$(*) \qquad \sum_{n=1}^{\infty} \frac{\|e_n\|_k}{\|e_n\|_{k+1}} < \infty \quad \text{for } k \in N.$$

This implies that $(A, \|\cdot\|_k)$ is a nuclear Fréchet algebra [1]. It is easy to check that for $0 < a < 1$ the matrix $a_{k,n} = a^{n/3^k}$, $k, n \in N$, satisfies conditions (a) and (b). Therefore, it determines a Fréchet algebra with unity and with the orthogonal basis.

1. Let $(A, \|\cdot\|_k)$ be a Fréchet algebra with unity and an orthogonal basis. In this section we shall find conditions implying the equality $A = s$ (in the sense of isomorphism). It is evident that the family of seminorms $(\|\cdot\|_k')$ defined by

$$\|x\|_k' = \sup_{p<q} \Big\| \sum_{n=p}^{q} e_n^*(x) e_n \Big\|_k$$

is equivalent to the family of seminorms $(\|\cdot\|_k)$; moreover, the following conditions are satisfied:

(1) $\|e_n\|_k' = \|e_n\|_k$ $(k, n \in N)$,

(2) $|e_n^*(x)| \, \|e_n\|_k' \leqslant \|x\|_k'$ $(x \in A, k \in N)$.

Without loss of generality we can assume that the family of seminorms $(\|\cdot\|_k)$ has properties (1) and (2).

THEOREM 1.1. *The following conditions are equivalent:*

(i) *For every $k \in N$ the set $\Delta_k = \{i \in N: \|e_i\|_k \neq 0\}$ is finite.*

(ii) $A = s$.

Proof. (i)$\Rightarrow$(ii). For every $k \in N$ there exists a positive integer n_k such that if $n \geqslant n_k$, then $\|e_n\|_k = 0$. Let

$$(t_n)_{n=1}^{\infty} \in s \quad \text{and} \quad x_n = \sum_{i=1}^{n} t_i e_i.$$

Then $\|x_m - x_n\|_k = 0$ for $m, n \geqslant n_k$. Thus $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(A, \|\cdot\|_k)$ and there exists $x_0 \in A$ such that

$$x_0 = \sum_{n=1}^{\infty} t_n e_n, \quad \text{i.e.} \quad (t_n)_{n=1}^{\infty} \in P.$$

(ii) $\Rightarrow$ (i). Suppose, on the contrary, that there exists a positive integer k_0 such that the set $\Delta_{k_0} = \{i \in N: \|e_i\|_{k_0} \neq 0\}$ is infinite. Let $i_1 < i_2 < \ldots < i_n < \ldots$, $i_n \in \Delta_{k_0}$, $n \in N$. We put $\alpha_n^{-1} = \|e_{i_n}\|_{k_0}$, $n \in N$. Therefore, for every $x \in A$ and $n \in N$ we have

$$|e_{i_n}^*(x)| \leqslant \alpha_n \|x\|_{k_0}.$$

Consider a sequence $(t_n)_{n=1}^{\infty} \in P$. Then $t_{i_n} = O(\alpha_n)$ and, consequently, $P \neq s$.

COROLLARY 1.1. *If $(A, \|\cdot\|_k)$ is an m-convex Fréchet algebra with unity and an orthogonal basis, then $A = s$.*

Proof. Since

$$e = \sum_{n=1}^{\infty} e_n,$$

we have $\|e_n\|_k \to 0$ for $k \in N$. Now fix k. Then there exists a positive integer n_k such that $\|e_n\|_k < 1$ for $n \geqslant n_k$, but then

$$\|e_n\|_k = \|e_n^r\|_k \leqslant \|e_n\|_k^r \to 0 \quad \text{as } r \to +\infty.$$

Hence for $n \geqslant n_k$ we have $\|e_n\|_k = 0$. Now apply Theorem 1.1.

An alternative proof of Corollary 1.1 was given in [3].

Definition 1.1. Let $(A, \|\cdot\|_k)$ be a Fréchet algebra with unity. We say that A has *Wiener's property* if it satisfies the condition

$$x \in A^{-1} \text{ iff } f(x) \neq 0 \text{ for every } f \in \mathfrak{M}(A),$$

where $A^{-1} = \{x \in A: x^{-1} \in A\}$.

THEOREM 1.2. *Let $(A, \|\cdot\|_k)$ be a Fréchet algebra with unity and the orthogonal basis $(e_n)_{n=1}^{\infty}$. If $(A, \|\cdot\|_k)$ has Wiener's property, then $A = s$.*

Proof. Suppose that $A \neq s$. Then there exist a sequence $(\alpha_n)_{n=1}^{\infty}$ with $\alpha_n > 0$ for $n \in N$ and an increasing sequence of indices $i_1 < i_2 < \ldots < i_n < \ldots$, $n \in N$, such that

(**) if $(t_n)_{n=1}^{\infty} \in P$, then $|t_{i_n}| = O(\alpha_n)$

(cf. the proof of Theorem 1.1). Let $t_1 > 0$, $t_n \leqslant t_{n+1}$ for $n \in N$, and $t_n \to +\infty$. In virtue of Theorem 0.1 we have $(t_n^{-1})_{n=1}^{\infty} \in P$. Put

$$x_0 = \sum_{n=1}^{\infty} t_n^{-1} e_n.$$

Then $e_n^*(x_0) = t_n^{-1} \neq 0$. Thus

$$x_0^{-1} = \sum_{n=1}^{\infty} t_n e_n \in A,$$

whence $(t_n)_{n=1}^{\infty} \in P$. Therefore, if $t_1 > 0$ and $(t_n)_{n=1}^{\infty}$ is an increasing sequence tending to $+\infty$, then $(t_n)_{n=1}^{\infty} \in P$. This contradicts (**) and the proof is completed.

Definition 1.2. Let $(X, \|\cdot\|_k)$ be a Fréchet space and let $x^* \in (X, \|\cdot\|_k)^*$. The number

$$r(x^*) = \inf\{k:\ x^* \text{ is continuous with respect to } \|\cdot\|_k\}$$

is called the *range of the functional* x^*.

In the proof of the next theorem we use the following

THEOREM (Eidelheit [2]). *Let* $x_n^* \in (X, \|\cdot\|_k)^*$ *and*

$$\lim_{n\to\infty} r(x_n^*) = +\infty.$$

For every sequence $(t_n)_{n=1}^{\infty} \in s$ *there exists* $x \in X$ *such that* $x_n^*(x) = t_n$ *for* $n \in N$.

THEOREM 1.3. *Let* $(A, \|\cdot\|_k)$ *be a Fréchet algebra with unity and the orthogonal basis* $(e_n)_{n=1}^{\infty}$. *If* $|e_n^*(x)| \leqslant \|x\|_{r(e_n^*)}$ *for every* $n \in N$ *and* $x \in A$, *then* $A = s$.

Proof. It is sufficient to show that $r(e_n^*) \to \infty$ and apply the theorem of Eidelheit. Assume that

$$\underline{\lim}_{n\to\infty} r(e_n^*) \neq +\infty.$$

Then there exists a positive integer k_0 such that the set

$$\Delta = \{i \in N:\ r(e_i^*) = k_0\}$$

is infinite. Hence for $i \in \Delta$ we have $1 = |e_i^*(e_i)| \leqslant \|e_i\|_{k_0}$. Therefore

$$\overline{\lim_{n\to\infty}} \|e_n\|_{k_0} \geqslant 1,$$

which is impossible and the proof is completed.

2. In this section we shall find conditions implying continuity of every multiplicative linear functional defined on A.

Let $(A, \|\cdot\|_k)$ be a Fréchet algebra with unity and the orthogonal basis $(e_n)_{n=1}^{\infty}$. We note that if f is a discontinuous multiplicative linear functional on A, $x, y \in A$, and $x-y \in \operatorname{lin}\{e_1, \ldots, e_n\}$, then $f(x) = f(y)$. This follows from the fact that $f(e_n) = 0$ for every $n \in N$.

THEOREM 2.1. *Let* $(A, \|\cdot\|_k)$ *be a Fréchet algebra with unity and the orthogonal basis* $(e_n)_{n=1}^{\infty}$. *If there exists a sequence* $(t_n)_{n=1}^{\infty} \in P$ *such that* $t_n \leqslant t_{n+1}$ *for* $n \in N$ *and* $t_n \to +\infty$, *then every multiplicative linear functional on* A *is continuous.*

Proof. Let $\mathfrak{N}(A)$ be the set of all non-trivial multiplicative linear functionals on A and $f \in \mathfrak{N}(A) \setminus \mathfrak{M}(A)$. Put

$$x_0 = \sum_{n=1}^{\infty} t_n e_n, \qquad x_0 \in A, f(x_0) = \alpha_0.$$

Of course, $\alpha_0 \neq 0$. We may assume that $t_n \neq \alpha_0$ for every $n \in N$. Let $v_n = t_n - \alpha_0$ for every $n \in N$. Then $v_n \neq 0$, $v_n \leqslant v_{n+1}$ for $n \in N$, and $v_n \to +\infty$. From Theorem 0.1 it follows that $(v_n^{-1})_{n=1}^{\infty} \in P$. Let

$$y_0^{-1} := \sum_{n=1}^{\infty} v_n^{-1} e_n.$$

Evidently, we have $y_0 = x_0 - \alpha_0 e$ and $f(y_0) = \alpha_0 - \alpha_0 f(e) = 0$, which is impossible since $y_0 \in A^{-1}$. This completes the proof.

THEOREM 2.2. *Let $(A, \|\cdot\|_k)$ be a Fréchet algebra with unity and the orthogonal basis $(e_n)_{n=1}^{\infty}$. If there exist a sequence $(t_n)_{n=1}^{\infty}$ and a number $s > 0$ such that $|t_n| > n^s$ for $n \in N$, then $\mathfrak{N}(A) = \mathfrak{M}(A)$.*

Proof. Let i_0 be a positive integer such that $s' = i_0 s > 1$. Let

$$x_0 = \sum_{n=1}^{\infty} t_n e_n.$$

Then $x_0^{i_0} = \sum_{n=1}^{\infty} t_n^{i_0} e_n \in A$; therefore

$$n^{s'} \|e_n\|_k \leqslant \|e_n\|_k |t_n^{i_0}| \leqslant \|x_0^{i_0}\|_k \quad \text{for } k, n \in N.$$

If $s'' > 0$ and $s' - s'' > 1$, then

$$n^{s''} \|e_n\|_k \leqslant \frac{1}{n^{s'-s''}} \|x_0^{i_0}\|_k \quad \text{for } k, n \in N.$$

Thus $(n^{s''})_{n=1}^{\infty} \in P$. Applying now Theorem 2.1 we complete the proof.

THEOREM 2.3. *Let $(A, \|\cdot\|_k)$ be a nuclear Fréchet algebra with unity and the orthogonal basis $(e_n)_{n=1}^{\infty}$. Then $\mathfrak{N}(A) = \mathfrak{M}(A)$.*

Proof. The basis $(e_n)_{n=1}^{\infty}$ is absolute (cf. [1]). Therefore

$$\sum_{n=1}^{\infty} \|e_n\|_k < \infty \quad \text{for } k \in N.$$

Then there exists a sequence $(t_n)_{n=1}^{\infty}$ such that $t_n > 0$ for $n \in N$, $t_n \to +\infty$, and

$$\sum_{n=1}^{\infty} t_n \|e_n\|_k < \infty \quad \text{for } k \in N.$$

Put $x_0 = \sum_{n=1}^{\infty} t_n e_n$. Then $\|e_n\|_k \leqslant t_n^{-1} \|x_0\|_k$ for $n, k \in N$ (cf. [4], p. 3) and there exists a sequence $(v_n)_{n=1}^{\infty}$ such that $0 < v_n \leqslant v_{n+1}$, $v_n \to +\infty$, and $v_n = O(t_n)$. Since

$$v_n \|e_n\|_k \leqslant \frac{v_n}{t_n} \|x_0\|_k \quad (k \in N),$$

we have $(v_n)_{n=1}^{\infty} \in P$. The proof is complete.

Now, we want to propose the problem the solution of which seems to be important for the development of the theory of algebras of this type. We begin with the following remark.

Let $(A, \|\cdot\|_k)$ be a Fréchet algebra with unity and the orthogonal basis $(e_n)_{n=1}^{\infty}$ and let $(t_n)_{n=1}^{\infty} \in P$, $\overline{\lim_{n\to\infty}} |t_n| = +\infty$. Let $k_1 < k_2 < \ldots < k_n < \ldots$ be an increasing sequence of positive integers such that $|t_{k_n}| \geq n^2$ for $n \in N$. Set $x_0 = \sum_{n=1}^{\infty} t_n e_n$. Then we have

$$\|e_{k_n}\|_i \leqslant n^{-2} \|x_0\|_i \qquad \text{for } i, n \in N.$$

Let $A_1 = \overline{\operatorname{lin}\{e_{k_1}, \ldots, e_{k_n}, \ldots\}}$.

It is clear that

(1) $(A_1, \|\cdot\|_k)$ is a Fréchet algebra with unity and the orthogonal basis $(e_{k_n})_{n=1}^{\infty}$ and $(\sqrt{n})_{n=1}^{\infty} \in P$.

(2) Every multiplicative linear functional on A_1 is continuous. Therefore, if f is a discontinuous multiplicative linear functional on A, then $f|A_1 = 0$.

Consequently, if $(t_n)_{n=1}^{\infty} \in P$ and $\overline{\lim_{n\to\infty}} |t_n| = +\infty$, then the set of all positive integers can be split into two disjoint subsets N_1 and N_2 such that putting

$$A_1 = \overline{\operatorname{lin}(e_i)_{i\in N_1}} \quad \text{and} \quad A_2 = \overline{\operatorname{lin}(e_i)_{i\in N_2}},$$

we have

(3) $A = A_1 \oplus A_2$.

(4) If f is a discontinuous multiplicative linear functional on A, then $f|A_1 = 0$.

Problem. Does there exist a maximal subalgebra $A_1' = \overline{\operatorname{lin}(e_i)_{i\in N_1'}}$ for which (3) and (4) hold? (**P 1274**)

Suppose that such an algebra A_1' exists and let $A_1' = \overline{\operatorname{lin}(e_i)_{i\in N_1'}}$. Then the set $N_2' = N \backslash N_1'$ is either empty or infinite. If $N_2' = \emptyset$, then every multiplicative linear functional on A is continuous. If $\bar{N}_2' = \aleph_0$, then writing $N_2' = \{k_n\}_{n=1}^{\infty}$, where $k_1 < k_2 < \ldots$, we can easily show that for every $x \in A$ the sequence $(e_{k_n}^*(x))_{n=1}^{\infty}$ is bounded.

Let $\varphi_0 \in \beta N \backslash N$, where βN is the Čech-Stone compactification of the set of all positive integers. Then the formula $f(x) = \varphi_0(e_{k_n}^*(x))_{n=1}^{\infty}$ $(x \in A)$ defines a non-trivial discontinuous multiplicative linear functional on A.

REFERENCES

[1] C. Bessaga and I. R. Retherford, *Lecture on nuclear spaces and related topics.*

[2] M. Eidelheit, *Zur Theorie der Systeme linearer Gleichungen*, Studia Mathematica 6 (1936), p. 139-148.

[3] T. Husain and J. Liang, *Multiplicative functionals on Fréchet algebras with bases*, Canadian Journal of Mathematics 29 (1977), p. 270-276.

[4] W. Żelazko, *Metric generalizations of Banach algebras*, Dissertationes Mathematicae 47 (1965).

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