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COVERING THEOREMS FOR UNIQUENESS AND EXTENDED UNIQUENESS SETS

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Introduction. A covering theorem for a class of sets $\mathscr{C}$ asserts that every set in $\mathscr{C}$ can be covered by a countable union of sets in some (somehow simpler) class $\mathscr{C}'$.

In the theory of sets of uniqueness on the unit circle $\boldsymbol{T}$ the first result of this kind is Piatetski-Shapiro's theorem in [PS], which states that every closed set of uniqueness can be covered by countably many closed sets in the class U_1', consisting of those closed sets $E \subseteq \boldsymbol{T}$ for which there exists a sequence of functions in $A(\boldsymbol{T})$, vanishing on E, which converges to the function 1 in the weak*-topology.

Recently, attention has been focused on such covering results for uniqueness sets ($\mathscr{U}$-sets) as well as extended uniqueness sets ($\mathscr{U}_0$-sets) in an attempt to overcome the difficulty of characterizing these sets, revealed by the theorem of Solovay [S] (see also [KL] and Kaufman [Ka2]) that the sets

$$U = \mathscr{U} \cap K(\boldsymbol{T}) \quad \text{and} \quad U_0 = \mathscr{U}_0 \cap K(\boldsymbol{T})$$

of closed $\mathscr{U}$- and $\mathscr{U}_0$-sets are coanalytic but not Borel subsets of the space $K(\boldsymbol{T})$ of closed subsets of $\boldsymbol{T}$. Perhaps there is a simply characterizable (in particular, Borel) class of U- (resp. U_0-) sets such that every U- (resp. U_0-) set is a countable union of sets in this class.

In 1986, we proved such a covering result for U_0, namely that every closed set of extended uniqueness can be written as a countable union of sets in the so-called class U_0', consisting of those closed sets E for which there is $a > 0$ such that, for every probability measure μ on E,

$$\overline{\lim}\, |\hat{\mu}(n)| \geqslant a.$$

Our original proof of this result has not been published and used a good deal of descriptive set theory.

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Using our result and some more descriptive set theory, especially results from [KLW] as well as a strengthening of the Kaufman–Solovay Theorem due to Kaufman [Ka3], Debs and Saint Raymond [DStR] extended this to Borel and even analytic sets in $\mathscr{U}_0$: Every analytic $\mathscr{U}_0$-set can be covered by countably many U'_0-sets. This gives a complete description of analytic $\mathscr{U}_0$-sets and has several consequences in the theory of sets of uniqueness, like the solution of the Category Problem (see [B1], and also [KL]): Every (extended) uniqueness set with the property of Baire is of the first category in $\boldsymbol{T}$.

In the first part of this paper, we give a direct new proof of these two covering results, using only tools from functional analysis and a Baire category-type argument. This direct approach has also the advantage to adapt to the case of sets of uniqueness (as opposed to extended uniqueness), where the situation is more complicated. The point here is that although U'_0 is a subclass of U_0, U'_1 is not a subclass of U, by Körner's Theorem on Helson sets of multiplicity [Kö]. In fact, in [DStR] Debs and Saint Raymond prove that there is no family $B \subseteq U$ which is Borel and forms a basis for U in the sense that every U-set can be covered by countably many sets in B or, equivalently, $U = B_\sigma$, which by definition is the σ-ideal of closed sets generated by B. On the other hand, U'_0 is Borel and a basis for U_0. (We prove also in [KL] that if $B \subseteq U$ is such that $U = B_\sigma$, then there is an extension B_1 of B with $B = B_1 \cap U$ and $(B_1)_\sigma = (U'_1)_\sigma$, i.e., B_1 generates the σ-ideal generated by U'_1, so that in some sense the only way to improve the Piatetski-Shapiro basis $U'_1 \cap U$ for U is to get a better basis for $(U'_1)_\sigma$ itself, not U.)

Despite these negative results, which prevent the descriptive set theoretic approach of [DStR] to be used in the context of U- and U_1-sets, the methods developed in Section 1 allow us to extend, in the second part of the paper, Piatetski-Shapiro's result to analytic $\mathscr{U}$-sets: Every analytic $\mathscr{U}$-set, in fact every analytic set which contains no closed set of multiplicity, can be covered by countably many sets in U'_1. We show moreover that this result is best possible: The σ-ideal $(U'_1)_\sigma \equiv U_1^*$ is the least σ-ideal of closed sets I with the property that every G_δ set which contains no closed set of multiplicity can be covered by countably many sets in I. Whereas U'_1 was defined indirectly by weakening the definition of U', the subclass of closed sets of uniqueness consisting of those E for which there is a sequence $f_n \in A(\boldsymbol{T})$ of functions vanishing in an open neighborhood (nbhd) of E which weak*-converges to 1, the above result gives surprisingly an "intrinsic" characterization of the σ-ideal U_1^* in terms of U. Another intrinsic characterization of U_1^* in terms of U has been recently obtained by Dougherty and Kechris [DK], who showed that U_1^* consists of exactly those closed sets within which, for some h, Hausdorff μ_h-measure 0 implies uniqueness.

Finally, we relate the class U_1^* with questions of harmonic synthesis of pseudomeasures. The class U_1^* can be characterized as consisting of those closed sets which support no pseudofunction satisfying synthesis or, equivalently, those closed sets for which uniqueness holds for all trigonometric series $\sum S(n)e^{inx}$ with S (viewed as a pseudomeasure) satisfying synthesis. Extending this definition to arbitrary subsets of T produces a new type of "uniqueness" set $\mathcal{U}_1^*$ lying strictly between $\mathcal{U}$ and $\mathcal{U}_0$ ($\mathcal{U} \subsetneqq \mathcal{U}_1^* \subsetneqq \mathcal{U}_0$). We show moreover that every set which contains no closed set of multiplicity is also contained in $\mathcal{U}_1^*$, i.e., it is a set of "uniqueness" as far as trigonometric series satisfying synthesis are concerned. (Whether it is actually a uniqueness set is the so-called Interior Problem — see [B1] and [KL] — and remains open.) Combining this with earlier results of Kechris and Louveau [KL] and independently Debs and Saint Raymond [DStR], one has a further corollary on the Union Problem for Borel $\mathcal{U}$-sets. The union of countably many $G_{\delta\sigma}$ $\mathcal{U}$-sets is of uniqueness for trigonometric series satisfying synthesis. So a possible counterexample to the Union Problem for G_δ sets has to incorporate a construction of non-synthesis sets (Malliavin's Theorem, see [KS] or [GMcG]); in fact, even more, the existence of sets in $U_1' - U$ (Körner's Theorem [Kö]).

In the sequel we use standard terminology and notation in harmonic analysis; see, e.g., [KS], [GMcG] or [KL].

1. The case of U_0-sets. For every closed set $E \subseteq T$, let us denote, respectively, by $M(E)$, $M^+(E)$, PROB(E) the spaces of (complex) measures, positive measures and probability measures supported by E.

A measure $\mu \in M(T)$ is a *Rajchman measure* if

$$\hat{\mu}(n) \to 0, \quad \text{i.e.,} \quad R(\mu) \stackrel{\text{def}}{=} \overline{\lim} |\hat{\mu}(n)| = 0.$$

We denote by $\mathcal{R}$ the set of Rajchman probability measures and for each $\varepsilon > 0$ by $\mathcal{R}_\varepsilon$ the set

$$\mathcal{R}_\varepsilon = \{\mu \in \text{PROB}(T)\colon\ R(\mu) < \varepsilon\},$$

so that

$$\mathcal{R} = \bigcap_{\varepsilon > 0} \mathcal{R}_\varepsilon.$$

A set $P \subseteq T$ is an *extended uniqueness set* (a *$\mathcal{U}_0$-set*) if, for every $\mu \in \mathcal{R}$, $\mu(P) = 0$. Otherwise, P is called of *restricted multiplicity* (an *$\mathcal{M}_0$-set*). As is well-known, if P is analytic, this is equivalent to

$$\mu(P) < 1 \quad \text{for all } \mu \in \mathcal{R}$$

or to

$$\forall \mu \in M(T) \big(\textstyle\sum \hat{\mu}(n) e^{int} = 0,\ \forall t \notin P \Rightarrow \mu = 0\big).$$

In particular, an analytic set P is in $\mathcal{U}_0$ iff every closed subset E of P is in $\mathcal{U}_0$. We denote by $K(T)$ the space of closed subsets of T and we let

$$U_0 = \mathcal{U}_0 \cap K(T)$$

be the family of *closed* $\mathscr{U}_0$*-sets*. Let also

$$M_0 = \mathscr{M}_0 \cap K(T).$$

Then for $E \in K(T)$

$$E \in U_0 \Leftrightarrow \mathrm{PROB}(E) \cap \mathscr{R} = \varnothing.$$

We define a subclass U'_0 of U_0 by

$$E \in U'_0 \Leftrightarrow \exists \varepsilon > 0 (\mathrm{PROB}(E) \cap \mathscr{R}_\varepsilon = \varnothing).$$

The family U'_0 is a Borel (in fact $G_{\delta\sigma}$) subset of $K(T)$, equipped with the standard Hausdorff metric under which it is compact. On the other hand, U_0 is co-analytic but not Borel ([S], [Ka2]; see also [KL]).

1.1. THEOREM. *The class U_0 is the σ-ideal generated by U'_0, i.e. every closed set of extended uniqueness is a countable union of sets in U'_0.*

Proof. Let $E \in K(T)$ be a set which cannot be covered by countably many sets in U'_0. We want to show that $E \notin U_0$. Let $F \subseteq E$ be defined by

$$x \in F \Leftrightarrow x \in E \,\&\, \forall V\ [V \text{ (open) nbhd of } x \Rightarrow E \cap V \text{ cannot be covered by countably many sets in } U'_0].$$

The set F is closed and non-empty by the hypothesis on E. Moreover, for any open V,

$$F \cap V \neq \varnothing \Rightarrow \overline{F \cap V} \notin U'_0.$$

We show now that

$$\mathscr{R}(F) = \mathscr{R} \cap \mathrm{PROB}(F)$$

is non-empty, a contradiction.

First we claim that for each $\varepsilon > 0$

$$\mathscr{R}_\varepsilon(F) = \mathscr{R}_\varepsilon \cap \mathrm{PROB}(F)$$

is dense in PROB(F) equipped with the usual w*-topology for which it is compact and metrizable. To see this, note that $\mathscr{R}_\varepsilon(F)$ is convex. Hence it is enough to show that the Dirac measures δ_x, $x \in F$, are in $\overline{\mathscr{R}_\varepsilon(F)}^{w^*}$. Let V_n be a sequence of nbhds of x with $\mathrm{diam}(V_n) \to 0$ and (as $\overline{V_n \cap F} \notin U'_0$) let

$$\mu_n \in \mathrm{PROB}(\overline{V_n \cap F})$$

be in $\mathscr{R}_\varepsilon(F)$. Clearly,

$$\mu_n \xrightarrow{w^*} \delta_x$$

and we are done.

We construct now inductively a sequence $\{\mu_n\}$ in PROB(F) and an increasing sequence $\{N_n\}$ of positive integers such that

(*) If $i \leqslant n$ and $N_i \leqslant |k| \leqslant N_{i+1}$, then

$$|\hat{\mu}_n(k)| < 2^{-i-1}.$$

Then if μ is a w*-limit of a subsequence of $\{\mu_n\}$, clearly

$$|\hat{\mu}(k)| \leqslant 2^{-i-1} \quad \text{for } |k| \geqslant N_i;$$

hence $\mu \in \mathscr{R}(F)$.

To construct $\{\mu_n\}$, $\{N_n\}$ we choose μ_n to satisfy (*) for $i \leqslant n-1$ and

(**) $\forall |k| \geqslant N_n$, $|\hat{\mu}_n(k)| \leqslant 2^{-n-4}$.

For $n = 0$, pick $\mu_0 \in \mathscr{R}_\varepsilon(f)$ with $\varepsilon = 2^{-4}$, and then N_0 so that

$$|\hat{\mu}_0(k)| \leqslant 2^{-4} \quad \text{for } |k| \geqslant N_0.$$

Suppose now that $\mu_0, \ldots, \mu_n, N_0, \ldots, N_n$ have been defined satisfying (*) and (**). Let m be any number $\geqslant N_n$. There are a measure $\mu(m)$ and an integer $\varphi(m) > m$ such that

(i) $\mu(m)$ satisfies (*) for $i \leqslant n-1$;

(ii) $|\widehat{\mu(m)}(k)| \leqslant 2^{-n-3}$ for $N_n \leqslant k < m$;

(iii) $|\widehat{\mu(m)}(k)| \leqslant 2^{-n-5}$ for $|k| \geqslant \varphi(m)$.

To see this, note that the measure μ_n satisfies (i), (ii), so by the density of $\mathscr{R}_\varepsilon(F)$, with $\varepsilon = 2^{-n-5}$, we can find $\mu(m) \in \mathscr{R}_\varepsilon(F)$ satisfying (i), (ii). Then choose $\varphi(m)$ to make (iii) true.

Define then a sequence $\{\nu_j\}$ in PROB(f) and $\{m_j\}$ by

$$\nu_0 = \mu(N_n), \quad m_0 = \varphi(N_n)$$

and

$$\nu_{j+1} = \mu(m_j), \quad m_{j+1} = \varphi(m_j).$$

Let for each k

$$\theta_k = \frac{1}{k+1}\sum_{j=0}^{k} \nu_j.$$

Then θ_k satisfies (*) for $i \leqslant n-1$ and

$$|\hat{\theta}_k(m)| \leqslant 2^{-n-5} \quad \text{for } |m| \geqslant m_k.$$

If now $N_n \leqslant |m| < m_k$, there is at most one j (the one for which $m_j \leqslant |m| < m_{j+1}$) such that $|\hat{\nu}_j(m)| \geqslant 2^{-n-3}$; hence

$$|\hat{\theta}_k(m)| \leqslant \frac{k \cdot 2^{-n-3} + 1}{k+1}.$$

Thus if we choose k large enough and let $\mu_{n+1} = \theta_k$, $N_{n+1} = m_k$, we also get (*) for $i = n$ and (**). This completes the proof.

This result leads to some kind of description of the σ-ideal of closed extended uniqueness sets, similar to the classical case of countable closed sets: Define

$$M_0^\xi = \{E \in K(T)\colon \forall \text{ open } V\ (V \cap E \neq \varnothing \Rightarrow \overline{V \cap E} \notin U_0')\},$$

which, by the preceding theorem, is equal to

$$\{E \in K(T)\colon \forall \text{ open } V\,(V \cap E \neq \emptyset \Rightarrow \overline{V \cap E} \notin U_0)\}.$$

Then any closed set E can be uniquely decomposed into a kernel E^*, the largest M_0^ξ set contained in E, and $E - E^*$ which is a countable union of U_0'-sets. And E^* can be obtained by a derivation process, similar to the Cantor–Bendixson one: for each closed set F, let

$$d(F) = \{x \in F\colon \forall \text{ open } V\,(x \in V \Rightarrow \overline{V \cap F} \notin U_0')\}.$$

Define inductively $E_0^{(\alpha)}$ by

$$E_0^{(0)} = E, \qquad E_0^{(\alpha+1)} = d(E_0^{(\alpha)})$$

and

$$E_0^{(\lambda)} = \bigcap_{\alpha<\lambda} E_0^{(\alpha)} \qquad \text{for limit } \lambda.$$

Then, for some $\alpha < \omega_1$, $E_0^{(\alpha+1)} = E_0^{(\alpha)} = E^*$. And as $E^* = \emptyset$ iff $E \in U_0$, this gives a countable "intrinsic" testing process for membership in U_0.

Note that although U_0 is complicated, i.e., co-analytic but not Borel, the class M_0^ξ (which is easily the same as the class of supports of Rajchman measures) is Borel, in fact $F_{\sigma\delta}$. This can be used to show for example that the perfect symmetric sets $E_{\varepsilon_1,\varepsilon_2,\dots}$ of varying ratios of dissection, which are in U_0, form a Borel ($G_{\delta\sigma}$) set in $K(T)$, suggesting perhaps the possibility of an "explicit" characterization of these sets (see [KL]).

We go now to the case of analytic sets in $\mathcal{U}_0$, i.e., the Debs–Saint Raymond Theorem. For simplicity let us consider first the case of G_δ sets.

1.2. THEOREM (Debs and Saint Raymond [DStR]). *Let H be a G_δ set in $\mathcal{U}_0$. Then H can be covered by countably many sets in U_0 (and hence in U_0' by Theorem 1.1).*

Equivalently, if H is a G_δ set which is dense in a non-empty M_0^ξ-set E, then H is not a $\mathcal{U}_0$-set.

Proof. The first assertion implies the second by the Baire Category Theorem and the fact that U_0-sets are nowhere dense in M_0^ξ-sets. Conversely, if H is a G_δ $\mathcal{U}_0$-set and cannot be covered by countably many U_0-sets, the set

$$H' = \{x \in H\colon \forall \text{ open } V\,(x \in V \Rightarrow V \cap H \text{ cannot be covered by countably many sets in } U_0)\}$$

is G_δ, non-empty and $E = \bar{H}'$ is in M_0^ξ, so that the second statement fails.

To prove the second assertion let

$$H = \bigcap_n V_n,$$

V_n open dense in E. Since $E \in M_0^\xi$, we infer that $\mathcal{R}(E)$ is w*-dense in PROB(E) (as in the proof of Theorem 1.1). Let

$$\mathcal{R}(V_n) = \mathcal{R}(E) \cap \{\mu\colon \operatorname{supp}(\mu) \subseteq V_n\}.$$

Since V_n is dense in E, $\mathscr{R}(V_n)$ is w*-dense in PROB(E), so $\mathscr{R}(V_n)$ is dense in PF $\cap$ PROB$(E)(=\mathscr{R}(E))$ for the weak topology of PF (= the space of pseudofunctions), hence for the norm topology of PF by Mazur's Theorem, since $\mathscr{R}(V_n)$ is convex.

Let $\mu_0 \in \mathscr{R}(E)$. As $\mathscr{R}(V_0)$ is dense in $\mathscr{R}(E)$, we can find W_1 open in E with

$$\bar{W}_1 \subseteq V_0, \quad \mu_1 \in \mathscr{R}(\bar{W}_1) \quad \text{and} \quad \|\mu_1 - \mu_0\|_{\mathrm{PF}} \leqslant 2^{-1}.$$

Now $\bar{W}_1 \in M_0^{\mathrm{g}}$ and $V_1 \cap W_1$ is open dense in it, so we can find W_2 open in E with

$$\bar{W}_2 \subseteq V_1 \cap W_1, \quad \mu_2 \in \mathscr{R}(\bar{W}_2) \quad \text{and} \quad \|\mu_1 - \mu_2\|_{\mathrm{PF}} \leqslant 2^{-2},$$

etc. So we construct a Cauchy sequence in PF of probability measures μ_n and a decreasing sequence W_n of open sets in E with

$$\bigcap W_n = \bigcap_n \bar{W}_n \subseteq H \quad \text{and} \quad \mu_n \in \mathscr{R}(\bar{W}_n).$$

So $\mu_n \to \mu \in \mathscr{R}$, $\mu \neq 0$, supp $(\mu) \subseteq \bigcap_n \bar{W}_n \subseteq H$, hence $H \notin \mathscr{U}_0$.

1.3. COROLLARY (Debs and Saint Raymond [DStR]). *Let $E \in M_0^{\mathrm{g}}$. If $A \subseteq E$ has the property of Baire in E and $A \in \mathscr{U}_0$, then A is of the first category in E. In particular, every $A \subseteq T$ with the property of Baire which is in $\mathscr{U}_0$ is of the first category.*

Proof. Since A has the property of Baire in E, $A = H \cup P$ with $H \in G_\delta$ and P meager in E. Since $H \in \mathscr{U}_0$, H can be covered by countably many (closed) U_0-sets, which, since $E \in M_0^{\mathrm{g}}$, are nowhere dense in E. Thus H is meager in E, and so is A.

1.4. COROLLARY (Menshov, see [B2]). *There is a (closed) M_0-set of (Lebesgue) measure* 0.

Proof. Let $G \subseteq T$ be a dense G_δ set of measure 0. Then $G \notin \mathscr{U}_0$, so there is a closed $E \subseteq G$, $E \in M_0$.

1.5. COROLLARY (Ivashev-Musatov [IM], Kaufman [Ka1]). *Let $E \in M_0$. Let h: $[0, \infty) \to [0, \infty)$ be non-decreasing with $h(0+) = 0$, $h(t) > 0$ for $t > 0$. Then there is a closed set $F \subseteq E$, $F \in M_0$ of h-Hausdorff measure* 0.

Proof. We can assume that $E \in M_0^{\mathrm{g}}$, $E \neq \varnothing$. Let $\{x_n\}$ be dense in E and choose, for each n, open intervals I_n^m with

$$x_m \in I_n^m \quad \text{and} \quad \sum_m h(|I_n^m|) \leqslant 1/n.$$

Then

$$H = \bigcap_n (E \cap \bigcup_m I_n^m)$$

is dense G_δ in E and of h-Hausdorff measure 0. But $H \notin \mathscr{U}_0$, hence H contains $F \in M_0$ which is still of h-Hausdorff measure 0.

1.6. COROLLARY (Kechris and Louveau [KL]). *Let $E \in M_0$ and let γ be* *a continuous capacity on $\boldsymbol{T}$ (i.e., a capacity such that $\gamma(K \cup \{x\}) = \gamma(K)$ for all $K \in K(\boldsymbol{T})$, $x \in \boldsymbol{T}$). Then there is a closed $F \subseteq E$ in M_0 with $\gamma(F) = 0$.*

Proof. As before, we may assume that $E \in M_0^\sigma$ and it is enough to construct a G_δ set H dense in E with $\gamma(H) = 0$. For that we construct as before open intervals I_n^m with

$$\gamma(\bar{I}_n^0 \cup \ldots \cup \bar{I}_n^m) < 1/n \qquad \text{for all } m,$$

using the continuity of γ.

1.7. Corollary (Lyons [L]). *The set $W^*(\{q^k\})$ of non-normal numbers to the base $q \geqslant 2$ is in $\mathscr{M}_0$.*

Proof. $W^*(\{q^k\})$ is the set of $x \in [0, 2\pi]$ for which the sequence $q^k x \pmod{2\pi}$ is not uniformly distributed, i.e., by Weyl's Criterion

$$\lim_N \frac{1}{N} \sum_{k=1}^{n} \exp\{iq^k mx\} \neq 0 \qquad \text{for some } m \in \boldsymbol{Z},\ m \neq 0.$$

Let

$$P = \left\{x \in [0, 2\pi] \colon \overline{\lim_N} \left| \frac{1}{N} \sum_{k=1}^{N} \exp\{iq^k x\} \right| = 1 \right\}.$$

The set P is comeager in $\boldsymbol{T}$ and $P \subseteq W^*(\{q^k\})$.

We prove now the extension of Theorem 1.2 to analytic sets.

1.8. Theorem (Debs and Saint Raymond [DStR]). *Let A be an analytic set in $\mathscr{U}_0$. Then A can be covered by countably many sets in U_0 (and thus in U_0').*

Proof. Assume this fails and let $H \subseteq \boldsymbol{T} \times 2^{\boldsymbol{N}}$ be G_δ with $A = \pi(H)$, where π denotes projection on $\boldsymbol{T}$. Let U_0^{ext} be the family of subsets of $\boldsymbol{T}$ which can be covered by countably many sets in U_0. Let

$$H' = \{x \in H \colon \forall \text{ open } V \text{ in } \boldsymbol{T} \times 2^{\boldsymbol{N}} \ (x \in V \Rightarrow \pi(V \cap H) \notin U_0^{\text{ext}})\}.$$

Then $H' \in G_\delta$ and, since $A \notin U_0^{\text{ext}}$, we have $H' \neq \emptyset$. Moreover, H' satisfies

$$\forall \text{ open } V \ (V \cap H' \neq \emptyset \Rightarrow \overline{\pi(V \cap H')} \notin U_0).$$

For F closed in $\boldsymbol{T} \times 2^{\boldsymbol{N}}$, let us say that F is "in M_0^σ" if, for all non-empty open V in F, $\pi \bar{V} \in M_0$. Thus $E = \bar{H}'$ is "in M_0^σ", and so is any $\bar{V}$ for V open non-empty in E. Note now that if F is "in M_0^σ" and $V \subseteq F$ is dense open in F, then

$$\mathscr{R}^\pi(F) = \{\mu \in \mathscr{R}(\pi F) \colon \exists K \subseteq V \ (K \text{ closed and } \operatorname{supp}(\mu) \subseteq \pi(K))\}$$

is, as in the proof of Theorem 1.2, w*-dense in $\mathrm{PROB}(\pi(F))$ and convex, hence norm-dense in $\mathscr{R}(\pi(F))$. Thus the same argument as in Theorem 1.2 allows us to build a measure $\mu \in \mathscr{R}(\pi(E))$ and a closed set $K \subseteq H'$ with $\operatorname{supp}(\mu) \subseteq \pi(H')$. Then $\pi(H') \notin \mathscr{U}_0$, so $A \notin \mathscr{U}_0$.

The argument behind Theorems 1.2 and 1.8 looks very much like a Baire category argument. In fact, a category-type largeness property of the class of Rajchman measures can be extracted from it.

Let us say that for $E \in K(T)$ a set $X \subseteq \mathrm{PROB}(E)$ is *almost comeager* if for any non-empty open set $V \subseteq \mathrm{PROB}(E)$ and any sequence $\{V_n\}$ of open dense in V convex sets in $\mathrm{PROB}(E)$ we have

$$X \cap (\bigcap_n V_n) \neq \varnothing.$$

The set of Rajchman probability measures on T is meager in $\mathrm{PROB}(T)$. But we have

1.9. THEOREM. *Let $E \in K(T)$ be non-empty in M_0^{ξ}. Then*

$$\mathscr{R}(E) = \mathscr{R} \cap \mathrm{PROB}(E)$$

is almost comeager in $\mathrm{PROB}(E)$.

Proof. Fix $\varnothing \neq V \subseteq \mathrm{PROB}(E)$, V convex open. Let also $G \subseteq V$ be convex, open and dense in V, and therefore in $Q = \bar{V}^{w^*}$. By the proof of Theorem 1.1, $\mathscr{R} \cap \mathrm{PROB}(E)$ is dense in $\mathrm{PROB}(E)$, so $\mathscr{R}_G = \mathscr{R} \cap G$ is dense in Q and of course convex. Now $\mathscr{R}_Q = \mathscr{R} \cap Q$ is a norm-closed subset of PF and

$$\bar{\mathscr{R}}_G^{w^*} \cap \mathrm{PF} = Q \cap \mathrm{PF} = R_Q.$$

Since the weak-closure $\bar{\mathscr{R}}_G^{w^*} \cap \mathrm{PF}$ of $\mathscr{R}_G$ in PF is the same as its norm-closure, it follows that $\mathscr{R}_G$ is norm-dense in $\mathscr{R}_Q$, and since $\mathscr{R}_G = G \cap \mathscr{R}_Q$, it is clearly open in $\mathscr{R}_Q$ with its strong topology.

To summarize: For every non-empty open convex $V \subseteq \mathrm{PROB}(E)$ and every open convex $G \subseteq V$ which is dense in V, $\mathscr{R}_G$ is open dense in $\mathscr{R}_Q$ in the norm-topology. So if $V \neq \varnothing$ is open in $\mathrm{PROB}(E)$ and V_n are open dense in V and convex, then assuming without loss of generality that V itself is convex (as it surely contains a non-empty open convex subset) we have, by the Baire Category Theorem in $\mathscr{R}_Q$, which being closed in PF is Polish,

$$\bigcap_n \mathscr{R}_{V_n} = (\bigcap_n V_n) \cap \mathscr{R}_Q \neq \varnothing$$

and we are done.

Note that Theorem 1.9 also easily implies Theorem 1.2: If the V_n's are decreasing open and dense in E, $E \in M_0^{\xi}$, $E \neq \varnothing$, then

$$V_n^* = \{\mu \in \mathrm{PROB}(E) \colon \mu(V_n) > \tfrac{1}{2}\}$$

is dense open and convex in $\mathrm{PROB}(E)$, so Theorem 1.9 gives a Rajchman probability measure μ with

$$\mu \in \bigcap_n V_n^*, \quad \text{i.e.,} \quad \mu(\bigcap_n V_n) \geqslant \tfrac{1}{2},$$

so that

$$\bigcap_n V_n \notin \mathscr{U}_0.$$

2. Results on U and U_1. A set $P \subseteq T$ is a *set of uniqueness* if every trigonometric series converging to 0 off P is identically 0. Denote by $\mathscr{U}$ the class of such sets and by $U = \mathscr{U} \cap K(T)$ the class of closed $\mathscr{U}$-sets. These can be also characterized as the closed sets $E \subseteq T$ for which the ideal

$$J(E) = \{f \in A(T)\colon f \text{ vanishes in a nbhd of } E\}$$

is w*-dense in $A(T)$ for the topology of duality with PF. We also let

$$U^{\text{int}} = \{P \subseteq T\colon \forall E \in K(T)(E \subseteq P \Rightarrow E \in U)\}$$

so that $\mathscr{U} \subseteq U^{\text{int}} \subsetneq \mathscr{U}_0$. Finally, $\mathscr{M}$, M are the classes of sets of *multiplicity* and closed sets of multiplicity, respectively.

Let, for $E \in K(T)$,

$$I(E) = \{f \in A(T)\colon f = 0 \text{ on } E\}.$$

A set $E \in K(T)$ is a U_1-*set* if $I(E)$ is w*-dense in $A(T)$, and a U'_1-*set* if $I(E)$ is w*-sequentially dense in $A(T)$, i.e., for some sequence $\{f_n\}$ in $I(E)$,

$$f_n \xrightarrow{w^*} 1.$$

We will also let

$$U_1^* = (U_1)_\sigma = (U'_1)_\sigma$$

be the σ-ideal of closed sets generated by U_1 or, equivalently, U'_1 by Piatetski-Shapiro's Theorem in [PS]. Finally, let

$$U_1^{\text{ext}} \ (=(U'_1)^{\text{ext}} = (U_1^*)^{\text{ext}})$$

be the σ-ideal of sets generated by U_1 or U'_1, i.e.,

$$U_1^{\text{ext}} = \{P \subseteq T\colon \exists \{E_n\} \in U_1 \ (P \subseteq \bigcup_n E_n)\}.$$

For a closed set E, we also let

$$\mathrm{PM}(E) = J(E)^\perp = \{S \in \mathrm{PM}\colon S \text{ is supported by } E\},$$

where PM is the space of pseudomeasures and

$$N(E) = I(E)^\perp = \{S \in \mathrm{PM}\colon \forall f \in I(E) \, (\langle f, S\rangle = 0)\}.$$

Clearly, $M(E) \subseteq N(E) \subseteq \mathrm{PM}(E)$ and $M(E)$ is w*-dense in $N(E)$ for the topology of duality with $A(T)$.

Piatetski-Shapiro's Theorem in [PS] implies, in particular, that $U \subseteq (U'_1)_\sigma$. Using the ideas of the first section we can extend this to Σ_1^1-sets.

2.1. THEOREM. *Let $P \subseteq T$ be Σ_1^1. If $P \in U^{\text{int}}$ (in particular, if $P \in \mathscr{U}$), then P can be covered by countably many sets in U'_1, i.e., $P \in U_1^{\text{ext}}$ $(= (U'_1)^{\text{ext}})$.*

We will prove this result in the case $P \in G_\delta$, the general case being handled by a projection argument, as in Theorem 1.8.

Let us denote by M_1^p the class of $E \in K(T)$ such that, for every non-empty open set V in E, $\bar{V} \notin U_1$. As usual, Theorem 2.1 is reduced to the following equivalent:

2.2. THEOREM. *If $E \in M_1^p$ and $H \subseteq E$ is G_δ dense in E, then H contains a set in M (i. e., $H \notin U^{\text{int}}$).*

We will need the following standard lemma:

2.3. LEMMA. *Let $x \in T$, $\{V_n\}$ open nbhds of x with* $\operatorname{diam}(V_n) \to 0$. *Let $\{S_n\}$ be pseudomeasures supported by $\bar{V}_n$ with*

$$S_n(0) = 1 \quad \textit{and} \quad \sup_n \|S_n\|_{\text{PM}} < \infty.$$

Then $S_n \xrightarrow{w^*} \delta_x$.

Proof. Let $f \in A(T)$ and $\varepsilon > 0$ be given. Let $\tilde{f}(t) = f(t) - f(x)$. Then, since $\tilde{f}(x) = 0$, we can find a $g \in A(T)$ with $\|g\|_A < \varepsilon$ and $g = \tilde{f}$ in a nbhd of x, hence in a nbhd of $\bar{V}_n$ for all large enough n. As $S_n \in \text{PM}(\bar{V}_n)$, it follows that for all large enough n

$$|\langle \tilde{f}, S_n \rangle| = |\langle g, S_n \rangle| \leqslant K\varepsilon \quad \text{with } K = \sup_n \|S_n\|_{\text{PM}}.$$

Since $\langle \tilde{f}, S_n \rangle = \langle f, S_n \rangle - \langle f, \delta_x \rangle$, we are done.

Let V be open in $E \in K(T)$. We define

$$\mathcal{V} = \{S \in \text{PF}\colon \exists K \in K(T)\ [K \subseteq V \,\&\, S \in N(K)]\}.$$

2.4. LEMMA. *Suppose that $E \in M_1^p$ and $V \subseteq E$ is open dense in E. Then $\mathcal{V}$ is* w**-dense in $N(E)$.*

Proof. It is enough to show that $\bar{\mathcal{V}}^{w^*}$ contains the measures with finite support in E. Since $\mathcal{V}$ is a subspace, it is enough to consider only Dirac measures δ_x, $x \in E$, and since V is dense in E, we can assume that $x \in V$. Let V_n be open nbhds of x in E with $\operatorname{diam}(V_n) \to 0$ and $\bar{V}_n \subseteq V$. As $E \in M_1^p$, there is

$$S_n \in N(\bar{V}_n) \cap \text{PF} \quad \text{with } S_n(0) = 1 \text{ and } \|S_n\|_{\text{PM}} < 2.$$

By Lemma 2.3,

$$S_n \xrightarrow{w^*} \delta_x$$

and we are done.

In particular, for E, V as in the lemma, $\mathcal{V}$ is dense in $N(E) \cap \text{PF}$ for the weak-topology of PF, hence in the norm-topology by Mazur's Theorem. We can proceed now exactly as in the proof of Theorem 1.2.

Remark. Colella [C] has also independently shown that if $E \in M_1^p$ and E_n is an increasing sequence of closed subsets of E with $E_n \in M_1^p$ and $\bigcup_n E_n$ dense in E, then for any $S \in \text{PF} \cap N(E)$ there are

$$S_n \in \text{PF} \cap N(E_n) \quad \text{with } \|S_n - S\|_{\text{PM}} \to 0.$$

We present now some applications.

Combining Theorem 2.2 with a result of Debs and Saint Raymond in

[DStR] we obtain first the following characterization of M_1^p and U_1^* in terms of U:

2.5. COROLLARY. *Let $E \in K(T)$. Then the following are equivalent*:

(i) $E \in M_1^p$.

(ii) *Every non-meager G_δ subset of E is not in U^{int}.*

In particular, U_1^* can be characterized as the least σ-ideal of closed sets I such that any G_δ set in U^{int} can be covered by countably many sets in I.

Proof. (i) $\Rightarrow$ (ii) is a restatement of Theorem 2.2: If H is non-meager G_δ in E, then $\bar{H}$ contains a non-empty set V open in E, and $\bar{V}$ is in M_1^p too, so that $H \cap \bar{V}$, and hence H, is not in U^{int}.

(ii) $\Rightarrow$ (i). In [DStR], the authors show that if $F \in U_1'$, then F contains a dense G_δ in U^{int}. Now, if $E \notin M_1^p$, then there is $\emptyset \neq V$ open in E with $F = \bar{V} \in U_1'$, so that there is G dense G_δ in F with $G \in U^{\mathrm{int}}$. Since G is a non-meager G_δ in E, we are done.

For the last assertion, first note that U_1^* satisfies this property by Theorem 2.1. If I is now any σ-ideal with this property and $E \in U_1^* - I$, towards a contradiction, there is $F \neq \emptyset$, $F \in U_1'$ which is I-perfect, i.e., $\bar{V} \notin I$ for all non-empty V open in F. Then subsets of F in I are nowhere dense in F, and so all G_δ sets in U^{int} which are contained in F are meager in F. This contradicts the result of Debs and Saint Raymond.

We also have analogs of Corollaries 1.4–1.6 with the same proofs.

2.6. COROLLARY. *Let $E \notin U_1^*$. Let γ be a continuous capacity or a Hausdorff measure. Then there is $F \subseteq E$, $F \in M$, with $\gamma(F) = 0$.*

It is not known if in the conclusion of this corollary one can get $F \notin U_1^*$. However, by a recent result of Dougherty and Kechris [DK] one cannot assume only that $E \in M$. In fact, they show that the following is true, thereby giving a different intrinsic characterization of U_1^* in terms of U: For $E \in K(T)$,

$$E \in U_1^* \Rightarrow \text{there is a Hausdorff measure } \mu_h \text{ on } T \text{ such that,} \quad \text{for all } F \subseteq E,\ F \in K(T),\ \mu_h(F) = 0 \Rightarrow F \in U.$$

Our next applications relate U_1^* to the Union and Interior Problems for sets of uniqueness, and problems of synthesis of pseudomeasures. (The Union Problem is the question of whether finite or countable unions of Borel sets in $\mathscr{U}$ are in $\mathscr{U}$ and the Interior Problem is the question of whether $(G_{\delta\sigma})$ $\mathscr{M}$-sets contain M-sets. The Union Problem is open even for two G_δ sets and the Interior Problem is open even for G_δ $\mathscr{M}$-sets.)

Let $\sum S(n)e^{inx}$ be a non-zero trigonometric series which converges to 0 almost everywhere, so that in particular $S \in \mathrm{PF}$. Let

$$\mathrm{RN}_S = \{x \in T\colon \sum_{|n| \leq N} S(n)e^{inx} \text{ is unbounded}\}$$

be the so-called *reduced nucleus* of S. It is an old problem of Bary [B1] to find out if RN_S is an $\mathscr{M}$-set. Recall that a pseudomeasure S *satisfies synthesis* if

$$f \in I(\operatorname{supp}(S)) \Rightarrow \langle f, S \rangle = 0.$$

2.7. COROLLARY. *The reduced nucleus of any S which satisfies synthesis is an $\mathscr{M}$-set, in fact it contains an M-set.*

Proof. If $S \in \mathrm{PF}$ satisfies synthesis, then $\operatorname{supp}(S) \in M_1^p$. But RN_S is a dense G_δ in $\operatorname{supp}(S)$, so we are done by Theorem 2.2.

2.8. COROLLARY. *Let G be a set in U^{int}. Then there is no non-zero trigonometric series $\sum S(n)e^{inx}$ with S satisfying synthesis such that*

$$\sum S(n)e^{inx} = 0, \ \forall x \notin G.$$

In particular, if $G_n \in U^{\mathrm{int}}$, $G_n \in \Sigma_3^0$ ($\equiv G_{\delta\sigma}$), then there is no such S with

$$\sum S(n)e^{inx} = 0, \ \forall x \notin \bigcup_n G_n.$$

Proof. If such an S exists, then $\mathrm{RN}_S \subseteq G$ and $\mathrm{RN}_S \notin U^{\mathrm{int}}$ by Corollary 2.7, so $G \notin U^{\mathrm{int}}$. For the second conclusion use the fact that U^{int} is closed under countable unions of $G_{\delta\sigma}$ sets (see [KL] and [DStR]).

This corollary says that any possible negative solution of the Union Problem for G_δ sets must incorporate a construction of non-synthesis sets, in fact sets in $U'_1 - U$ (Körner's Theorem [Kö]).

The concept of pseudomeasures satisfying synthesis is actually very closely connected to the concept of U_1^*-set. To see this we need the following lemma:

2.9. LEMMA. *Let $E \in K(T)$. Then the following are equivalent*:

(i) $E \in M_1^p$.

(ii) $E = \operatorname{supp}(S)$, *where* $S \in \mathrm{PF}$ *satisfies synthesis.*

Proof. The direction (ii) $\Rightarrow$ (i) is easy. So, conversely, let $E \in M_1^p$. Let $\{V_n\}$ enumerate all basic open sets with $V_n \cap E \neq \emptyset$. Then for each n there is a non-zero pseudofunction in $N(E)$ with support contained in V_n. We will define inductively on n a pseudofunction $S_n \in N(E)$ with $\|S_n\|_{\mathrm{PM}} = 1$, a function $f_n \in A$ with support contained in V_n and a sequence of positive numbers $\varepsilon_{n+1}^{(n)}, \varepsilon_{n+2}^{(n)}, \ldots$ as follows: For $n = 0$, let $S_0 \in \mathrm{PF}$ be such that

$$\|S_0\|_{\mathrm{PM}} = 1, \quad \operatorname{supp}(S_0) \subseteq V_0 \quad \text{and} \quad S_0 \in N(E).$$

Let $f_0 \in A$ be such that

$$\operatorname{supp}(f_0) \subseteq V_0 \quad \text{and} \quad \langle f_0, S_0 \rangle \neq 0.$$

Choose positive $\varepsilon_1^{(0)}, \varepsilon_2^{(0)}, \ldots$ so that

$$\|f_0\|_A \Big(\sum_{m \geq 1} \varepsilon_m^{(0)} \Big) < |\langle f_0, S_0 \rangle|.$$

Then for every $S'_1, S'_2, \ldots$ with $\|S'_m\|_{\mathrm{PM}} = 1$ and every $0 < \delta_m \leqslant \varepsilon_m^{(0)}$ we have

$$\langle f_0, \sum_{m \geqslant 1} \delta_m S'_m \rangle \neq 0.$$

Let now $\delta_1 = \varepsilon_1^{(0)}$ and find $S_1 \in \mathrm{PF}, f \in A$ such that

$$\|S_1\|_{\mathrm{PM}} = 1, \quad \mathrm{supp}(S_1) \subseteq V_1, \quad S_1 \in N(E), \quad \mathrm{supp}(f_1) \subseteq V_2$$

and

$$\langle f_1, S_0 + \delta_1 S_1 \rangle \neq 0.$$

Choose positive $\varepsilon_2^{(1)}, \varepsilon_3^{(1)}, \ldots$ so that

$$\|f_1\|_A \left(\sum_{m \geqslant 2} \varepsilon_m^{(1)} \right) < |\langle f_1, S_0 + \delta_1 S_1 \rangle|.$$

Then for every $S'_2, S'_3, \ldots$ with $\|S'_m\|_{\mathrm{PM}} = 1$ and every $0 < \delta_m \leqslant \varepsilon_m^{(1)}$ we have

$$\langle f_1, S_0 + \delta_1 S_1 + \sum_{m \geqslant 2} \delta_m S'_m \rangle \neq 0.$$

Let now $\delta_2 = \min\{\varepsilon_2^{(0)}, \varepsilon_2^{(1)}\}$ and proceed as before with V_2, etc. Letting now, in general,

$$\delta_n = \min\{\varepsilon_n^{(0)}, \varepsilon_n^{(1)}, \ldots, \varepsilon_n^{(n-1)}\},$$

we have $\sum \delta_n \leqslant \sum \varepsilon_n^{(0)} < \infty$, and thus if

$$S = S_0 + \sum_{n \geqslant 1} \delta_n S_n,$$

then $S \in \mathrm{PF}$ and $S \in N(E)$. Finally, $\mathrm{supp}(S) = E$, since $\langle f_n, S \rangle \neq 0$ for all n.

We have now

2.10. COROLLARY. *Let $E \in K(T)$. Then the following are equivalent*:

(i) $E \in U_1^*$.

(ii) *E supports no non-zero pseudofunction which satisfies synthesis.*

(iii) *Every trigonometric series $\sum S(n)e^{inx}$ for which S satisfies synthesis and*

$$\sum S(n)e^{inx} = 0, \ \forall x \notin E,$$

is identically 0.

Thus U_1^* appears as an interesting analog of U and U_0. Moreover, U_1^* has a natural extension to arbitrary subsets of T, namely if

$$P \in \mathscr{U}_1^* \Leftrightarrow \text{every trigonometric series } \sum S(n)e^{inx} \text{ for which } S \text{ satisfies synthesis and } \sum S(n)e^{inx} = 0, \ \forall x \notin E, \text{ is identically } 0,$$

then $\mathscr{U}_1^* \cap K(T) = U_1^*$. Note that, by Corollary 2.8, $U^{\mathrm{int}} \subseteq \mathscr{U}_1^*$.

A very interesting question is whether the conclusion of Theorem 2.1 goes through only under the hypothesis that $P \in (U_1^*)^{\mathrm{int}}$, thus obtaining a full extension of the Piatetski-Shapiro Theorem ($U_1 \subseteq (U'_1)_\sigma$). If so, then one would have for Σ_1^1-sets

$$\mathscr{U}_1^* = (U_1^*)^{\mathrm{int}} = (U_1^*)^{\mathrm{ext}}.$$

We finish by using the preceding ideas to provide a new proof of the Piatetski-Shapiro Theorem itself.

2.11. THEOREM (Piatetski-Shapiro [PS]). *Every U_1-set is a countable union of U'_1-sets.*

Proof. As usual it is enough to prove that if $\emptyset \neq E \in K(T)$ is such that for all non-empty open V in E, $\bar{V} \notin U'_1$, then $E \in M_1$. For $\varepsilon > 0$, let

$$N^{\varepsilon}(E) = \{S \in N(E)\colon \overline{\lim} |S(k)| < \varepsilon\}.$$

First note that $N^{\varepsilon}(E)$ is w*-dense in $N(E)$: To see this let

$$\mu = \sum_{i=1}^{k} \alpha_i \delta_{x_i}, \qquad x_i \in E,$$

be a finite support measure. Let $x_i \in V_n^i$, V_n^i open in E, $\operatorname{diam}(V_n^i) \underset{n}{\to} 0$. Choose then

$$S_n^i \in N(\bar{V}_n^i) \quad \text{with } R(S_n^i) < (\varepsilon/k) \max_i |\alpha_i|,\ \|S_n^i\|_{\mathrm{PM}} < 2,\ S_n^i(0) = 1.$$

Then

$$S_n^i \overset{w^*}{\underset{n}{\to}} \delta_{x_i},$$

and so if $S_n = \sum \alpha_i S_n^i$, we have

$$R(S_n) < \varepsilon \quad \text{and} \quad S_n \overset{w^*}{\to} \mu.$$

Next we need a lemma.

2.12. LEMMA. *Suppose that*

$$C \subseteq \mathrm{PM}^{\varepsilon} = \{S \in \mathrm{PM}\colon \overline{\lim} |\hat{S}(k)| < \varepsilon\}$$

is convex and $T \in \bar{C}^{w^*}$. Then there is a sequence $T_n \in C$,

$$T_n \overset{w^*}{\to} T \quad \textit{with } \|T_n\|_{\mathrm{PM}} < \|T\|_{\mathrm{PM}} + \varepsilon.$$

Proof. Let

$$C^{(1)} = \{S \in \mathrm{PM}\colon S \text{ is a w*-limit of a sequence in } C\}.$$

If we can prove the lemma for $T \in C^{(1)}$, we are done, since this implies that $C^{(1)}$ is closed under w*-limits of sequences, hence $C^{(1)} = \bar{C}^{w^*}$. So let $T \in C^{(1)}$ and let $S_n \in C$ be a sequence w*-converging to T, with say $\|S_n\|_{\mathrm{PM}} \leqslant M$. Given any N and $\delta < \varepsilon$ we will find $S \in C$ with

$$|T(m) - S(m)| < \delta \text{ for } |m| \leqslant N \quad \text{and} \quad \|S\|_{\mathrm{PM}} < \|T\|_{\mathrm{PM}} + \varepsilon.$$

For this note that for $p \geqslant N$ there are $S \in C$ with $\|S\|_{\mathrm{PM}} \leqslant M$ and $q \geqslant p$ such that

$$|T(m) - S(m)| < \delta \text{ for } |m| \leqslant p \quad \text{and} \quad |S(m)| < \varepsilon \text{ for } |m| \geqslant q.$$

By iterating and averaging as in the proof of Theorem 1.1 we are done.

Returning to the proof of Theorem 2.11, we see that for any $S \in N(E)$ with $\|S\|_{\mathrm{PM}} < \alpha$ there is a sequence $T_n \in N^{\varepsilon}(E)$ with

$$T_n \xrightarrow{w^*} S \quad \text{and} \quad \|T_n\|_{\mathrm{PM}} < \alpha + \varepsilon.$$

We define then inductively $S_1, S_2, \ldots$ in $N(E)$, and $0 < n_1 < n_2 < \ldots$, such that

$$\|S_1\|_{\mathrm{PM}} < 1, \quad \|S_k\|_{\mathrm{PM}} < 1 + \sum_{i \leqslant k} 2^{-i} \text{ for } k \geqslant 2,$$

$$|S_k(n)| < 2^{-i} \text{ for } |n| > n_i \text{ and } k \geqslant i \quad \text{and} \quad S_k(0) > \tfrac{1}{2}.$$

Then if $\{S_{k_i}\}$ is a w*-converging subsequence, say with limit S, we have $S \in N(E)$, $S \neq \varnothing$, $S \in \mathrm{PF}$, so that $E \in M_1$.

To define the S_i, n_i, choose first $S_1 \in N^{1/2}(E)$ with $\|S_1\|_{\mathrm{PM}} < 1$ and $S_1(0) > \frac{1}{2}$, and then n_1 with $|\hat{S}(n)| < \frac{1}{2}$ for $|n| \geqslant n_1$. By using the above, for each $m \geqslant n_1$ there are $m' \geqslant m$ and $S \in N(E)$ with

$$\|S\|_{\mathrm{PM}} \leqslant 1 + \tfrac{1}{2}, \quad |S(k)| < \tfrac{1}{2} \text{ for } n_1 < |k| < m,$$

$$|S(k)| < \tfrac{1}{4} \text{ for } |k| > m' \quad \text{and} \quad S(0) > \tfrac{1}{2}.$$

Iterating and averaging as before, we get S_2, and then n_2, etc.

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