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# ON SOME ADJUNCTIONS BETWEEN THE CATEGORIES OF ADJUNCTION-MORPHISMS AND MONAD-MORPHISMS

BY

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In this paper we consider categories whose objects are functors between the categories over a universe $U$. Following [3] and [5], we call them *2-categories* with respect to $U$. The category 2-CAT$(U)$, which consists of all categories of functors with natural transformations between the categories over $U$, i.e. objects of CAT$(U)$, may be regarded as an example of 2-category. Another example is 2-Cat$(U)$, consisting of all functor categories with natural transformations between the small categories over $U$, i.e. objects of Cat$(U)$. These two 2-categories are called *fundamental* with respect to $U$.

If $A$ is any 2-category with respect to $U$ (the symbol $U$ will be omitted whenever the universe is fixed), then $\mathrm{Adj}(A)$ (respectively, $\mathrm{Adj}^{\leftarrow}(A)$) is the category whose objects are adjoint pairs

$$\cdot \underset{g}{\overset{f}{\rightleftarrows}} \cdot$$

of functors with $f, g \in A$, with morphisms defined as in the category $d_{(1,1)}A$ (respectively, $d_{(3,3)}A$), given in [5]. The *objects* of $d_{(1,1)}A$ (respectively, $d_{(3,3)}A$) are all pairs

$$\cdot \underset{g}{\overset{f}{\rightleftarrows}} \cdot$$

of functors in $A$. Thus there is a forgetful functor from $\mathrm{Adj}(A)$ (respectively, $\mathrm{Adj}^{\leftarrow}(A)$) to $d_{(1,1)}A$ (respectively, $d_{(3,3)}A$). There is also a functor $F\colon \mathrm{Adj}(A) \to \mathrm{Mon}(A)$ (respectively, $F^{\leftarrow}\colon \mathrm{Adj}^{\leftarrow}(A) \to \mathrm{Mon}^{\leftarrow}(A)$) defined in a natural way to the category of monads in $A$. We do not know whether there exists any functor adjoint to $F$ (respectively, $F^{\leftarrow}$). In particular, we do not decide whether the functor $K\colon \mathrm{Mon}(A) \to \mathrm{Adj}(A)$ (respectively, $K^{\leftarrow}\colon \mathrm{Mon}^{\leftarrow}(A) \to \mathrm{Adj}^{\leftarrow}(A)$), defined in 1.6 for $A = $ 2-CAT$(U)$, is adjoint to $F$ (respectively, $F^{\leftarrow}$).

The purpose of this paper is to find the adjoint functors between categories in 2-Adj$(A)$ (respectively, 2-Adj$^{\leftarrow}(A)$) and 2-Mon$(A)$ (respectively, 2-Mon$^{\leftarrow}(A)$). These 2-categories were defined in [5]. We remind their definitions in Section 1. We determine the adjoint functors between categories in 2-Adj$(A)$ (respectively, 2-Adj$^{\leftarrow}(A)$) and 2-Mon$(A)$ (respectively, 2-Mon$^{\leftarrow}(A)$) in Section 2 in Theorem 2.1 (respectively, Theorem 2.2) in the special case $A = 2\text{-CAT}(U)$, leaving the general problem of finding such pairs of adjoint functors as an open question. (**P 1178**)

Another purpose of the paper is to determine isomorphisms between the categories in 2-Mon$^{\leftarrow}(A)$ and 2-Kan$(A{\downarrow}A)$ (the category Kan$(A{\downarrow}A)$ is due to Dubuc [2], and 2-Kan$(A{\downarrow}A)$ is obtained from Kan$(A{\downarrow}A)$ by using a scheme from [5]). We generalize a result of Alagic [1] and obtain, in Section 3, some characterization of so-called monadic functors.

The paper is a continuation of paper [5] by the second-named author. The reader is assumed to be familiar with that paper.

## 1. Preliminaries.

**1.1.** Let $U$ be a fixed universe. A *2-category* $A$ with respect to $U$ is a category $A_0$ (called the *local discrete category* of $A$) together with a family $\{A(X, Y): X, Y \text{ are objects of } A_0\}$ of categories $A(X, Y)$ such that

(i) $A_0$ is a subcategory of CAT$(U)$ or $A_0$ is an object of CAT$(U)$;

(ii) for any objects $X, Y$ in $A_0$ the objects of $A(X, Y)$ are all morphisms in $A_0$ from $X$ to $Y$, and for any objects $f, g$ in $A(X, Y)$ the set of all morphisms in $A(X, Y)$ from $f$ to $g$ is a subset of $U$;

(iii) for all objects $X, Y, Z$ in $A_0$ the rules of composition

$$A(X, Y)\times A(Y, Z) \xrightarrow{\circ=\circ_{X,Y,Z}} A(X, Z)$$

are associative functors which are unitary in all variables and agree with composition in $A_0$ on objects.

A 2-category $A$ with respect to $U$ is said to be *small* (or *over* $U$) provided $A_0$ is an object of CAT$(U)$ and $A(X, Y)$ is an object of Cat$(U)$ for any objects $X, Y$ in $A_0$. Let $A$ be a 2-category. Then, for all objects $X, Z$ in $A_0$, $1_X$ is the unit with respect to $\circ_{X,X,Z}$ and $\circ_{Z,X,X}$. The composition $\circ$ from (iii) is said to be *strong* in $A$, and the value of $\circ(\alpha, \beta)$ is denoted by $\beta\circ\alpha$ or by $\beta\alpha$. For all objects $X, Y$ in $A_0$ the composition in $A(X, Y)$ is said to be *weak* in $A$ and is denoted by $\cdot$. The objects and morphisms of $A_0$ are called *0-cells* and *1-cells* (or *functors*), respectively, in $A$.

**1.2.** Let $A$ be any 2-category with respect to $U$. By [5], the *objects* of the category $d_{(1,1)}A$ (respectively, $d_{(3,3)}A$) are all pairs $(f, g)$ of 1-cells in $A$ with codom$f$ = dom$g$. The *morphisms* in $d_{(1,1)}A$ (respectively, $d_{(3,3)}A$) from $(f\colon X\to X', g\colon X'\to X'')$ to $(f'\colon Y\to Y', g'\colon Y'\to Y'')$

are all pairs $(\langle h_1, h_2, \varphi_1\rangle, \langle h_2, h_3, \varphi_2\rangle)$ such that

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\varphi_1 |"] & X' \arrow[r, "g"] \arrow[d, "h_2"] \arrow[dr, phantom, "\varphi_2 |"] & X'' \arrow[d, "h_3"] \\
Y \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Y''
\end{tikzcd}$$

are diagrams in $A$, where $|$ is $\downarrow$ (respectively, $\uparrow$) for $d_{(1,1)}A$ (respectively, $d_{(3,3)}A$), and

$$\begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\varphi\downarrow"] & \cdot \arrow[d, "h_2"] \\
\cdot \arrow[r, "f'"'] & \cdot
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\varphi\uparrow"] & \cdot \arrow[d, "h_2"] \\
\cdot \arrow[r, "f'"'] & \cdot
\end{tikzcd}$$

are written instead of 2-cells (i.e. morphisms between 1-cells)

$$\varphi\colon h_2 f \to f'h_1 \quad \text{and} \quad \varphi\colon f'h_1 \to h_2 f,$$

respectively, in $A$.

The identity defined by the object $(f\colon X \to X',\ g\colon X' \to X'')$ is of the form $(\langle X, X', f\rangle, \langle X', X'', g\rangle)$, and the composition of morphisms in $d_{(1,1)}A$ and $d_{(3,3)}A$ is defined by the formula

$$\begin{aligned}
&(\langle h_1', h_2', \varphi_1'\rangle, \langle h_2', h_3', \varphi_2'\rangle)(\langle h_1, h_2, \varphi_1\rangle, \langle h_2, h_3, \varphi_2\rangle) \\
&\qquad\qquad = (\langle h_1'h_1, h_2'h_2, \varphi_1' \square \varphi_1\rangle, \langle h_2'h_2, h_3'h_3, \varphi_2' \square \varphi_2\rangle),
\end{aligned}$$

where, for $i = 1, 2$,

$$\varphi_i' \square \varphi_i = \begin{cases} \varphi_i' h_i \cdot h_{i+1}'\varphi_i & \text{for } d_{(1,1)}A, \\ h_{i+1}'\varphi_i \cdot \varphi_i' h_i & \text{for } d_{(3,3)}A. \end{cases}$$

By [5], the category $d_{(1,1)}A$ (respectively, $d_{(3,3)}A$) is the local discrete category of a 2-category $2\text{-}d_{(1,1)}A$ (respectively, $2\text{-}d_{(3,3)}A$) such that for any objects $a = (f, g)$ and $a' = (f', g')$ in $d_{(1,1)}A$ (respectively, $d_{(3,3)}A$) the morphisms in the category $2\text{-}d_{(1,1)}A(a, a')$ (respectively, $2\text{-}d_{(3,3)}A(a, a')$) from $(\langle h_1, h_2, \varphi_1\rangle, \langle h_2, h_3, \varphi_2\rangle)$ to $(\langle h_1', h_2', \varphi_1'\rangle, \langle h_2', h_3', \varphi_2'\rangle)$ are all pairs $(\langle \alpha, \beta\rangle, \langle \beta, \gamma\rangle)$ of 2-cells

$$\alpha\colon h_1 \to h_1', \quad \beta\colon h_2 \to h_2', \quad \gamma\colon h_3 \to h_3'$$

in $A$ such that

$$f'\alpha \cdot \varphi_1 = \varphi_1' \cdot \beta f, \quad g'\beta \cdot \varphi_2 = \varphi_2' \cdot \gamma g$$

and, respectively,

$$\beta f \cdot \varphi_1 = \varphi_1' \cdot f'\alpha, \quad \gamma g \cdot \varphi_2 = \varphi_2' \cdot g'\beta.$$

The *composition of morphisms* in these categories is given by the formula

$$(\langle \alpha_1, \beta_1\rangle, \langle \beta_1, \gamma_1\rangle) \cdot (\langle \alpha, \beta\rangle, \langle \beta, \gamma\rangle) = (\langle \alpha_1 \cdot \alpha, \beta_1 \cdot \beta\rangle, \langle \beta_1 \cdot \beta, \gamma_1 \cdot \gamma\rangle).$$

The *strong composition* $\circ$ in $2\text{-}d_{(i,i)}A$ $(i = 1, 3)$ is defined by

$$(\langle a', \beta'\rangle, \langle\beta', \gamma'\rangle)\circ(\langle a, \beta\rangle, \langle\beta, \gamma\rangle) = (\langle a'a, \beta'\beta\rangle, \langle\beta'\beta, \gamma'\gamma\rangle).$$

**1.3.** Let $A$ be any 2-category with respect to $U$. Following [5], we define

$$dA = \mathrm{pr}_1\, d_{(1,1)}A, \quad d^{\leftarrow}A = \mathrm{pr}_1\, d_{(3,3)}A,$$
$$2\text{-}dA = \mathrm{pr}_1\, 2\text{-}d_{(1,1)}A, \quad 2\text{-}d^{\leftarrow}A = \mathrm{pr}_1\, 2\text{-}d_{(3,3)}A.$$

By omitting the symbols concerning the second axis in the definitions of the categories on the right-hand side (see 1.2), we obtain the categories on the left-hand side of the above equalities.

A *monad* in $A$ is any 4-tuple $\boldsymbol{T} = \langle X, T, \eta, \mu\rangle$ consisting of a 1-cell $F'\boldsymbol{T} = T\colon X \to X$ in $A$ and 2-cells $\eta\colon X \to T$ and $\mu\colon T^2 \to T$ in $A$ such that

$$\mu\cdot T\mu = \mu\cdot\mu T \quad \text{and} \quad \mu\cdot T\eta = \mu\cdot\eta T = T.$$

Let $\boldsymbol{T} = \langle X, T, \eta, \mu\rangle$ and $\boldsymbol{T}' = \langle X', T', \eta', \mu'\rangle$ be two monads in $A$. Following [5], a *monad morphism* (respectively, *inverse monad morphism*) from $\boldsymbol{T}$ to $\boldsymbol{T}'$ is any morphism $\langle h_1, h_2, \varphi_1\rangle$ in $dA$ (respectively, $d^{\leftarrow}A$) from $F'\boldsymbol{T}$ to $F'\boldsymbol{T}'$ such that

$$h_1 = h_2, \quad \eta' h_1 = \varphi_1\cdot h_1\eta, \quad \varphi_1\cdot h_1\mu = \mu' h_1\cdot T'\varphi_1\cdot\varphi_1 T$$

and, respectively,

$$h_1 = h_2, \quad h_1\eta = \varphi_1\cdot\eta' h_1, \quad \varphi_1\cdot\mu' h_1 = h_1\mu\cdot\varphi_1 T\cdot T'\varphi_1.$$

The composition of monad morphisms (respectively, inverse monad morphisms) in $A$, which agrees with the composition in $dA$ (respectively, $d^{\leftarrow}A$), is a monad morphism (respectively, inverse monad morphism). Thus we obtain the monad category $\mathrm{Mon}(A)$ (respectively, $\mathrm{Mon}^{\leftarrow}(A)$), given in [5], whose objects and morphisms are monads and monad morphisms (respectively, inverse monad morphisms) in $A$. $\mathrm{Mon}(A)$ and $\mathrm{Mon}^{\leftarrow}(A)$ are the local discrete categories of 2-categories $2\text{-}\mathrm{Mon}(A)$ and $2\text{-}\mathrm{Mon}^{\leftarrow}(A)$ such that for any monads $\boldsymbol{T}, \boldsymbol{T}'$ in $A$ the category $2\text{-}\mathrm{Mon}(A)(\boldsymbol{T}, \boldsymbol{T}')$ (respectively, $2\text{-}\mathrm{Mon}^{\leftarrow}(A)(\boldsymbol{T}, \boldsymbol{T}')$) is a subcategory of the category $2\text{-}dA(F'\boldsymbol{T}, F'\boldsymbol{T}')$ (respectively, $2\text{-}d^{\leftarrow}A(F'\boldsymbol{T}, F'\boldsymbol{T}')$) determined by all monad morphisms (respectively, inverse monad morphisms) from $\boldsymbol{T}$ to $\boldsymbol{T}'$ and 2-cells of the form $\langle a, \beta\rangle = a$ with $a = \beta$. The strong compositions in $2\text{-}\mathrm{Mon}(A)$ (respectively, $2\text{-}\mathrm{Mon}^{\leftarrow}(A)$) agree with the strong compositions in $2\text{-}dA$ (respectively, $2\text{-}d^{\leftarrow}A$).

The *category* $A{\downarrow}A$ is a subcategory of $dA$ defined by all objects and all morphisms $\langle h_1, h_2, \varphi\rangle$ with $\varphi = \mathrm{id}$. The *category* $\mathrm{Kan}(A{\downarrow}A)$ is a full subcategory of $A{\downarrow}A$ determined by all objects $g$ having the right Kan extension of $g$ along $g$ which is preserved by 1-cells (see [2] and [5]). $\mathrm{Kan}(A{\downarrow}A)$ is a local discrete category of the 2-category $2\text{-}\mathrm{Kan}(A{\downarrow}A)$ such that for any objects $g$ and $g'$ the category $2\text{-}\mathrm{Kan}(A{\downarrow}A)(g, g')$ is the full subcategory of $2\text{-}dA(g, g')$ determined by all morphisms in $\mathrm{Kan}(A{\downarrow}A)$

from $g$ to $g'$. The strong compositions in 2-Kan$(A\downarrow A)$ agree with the strong compositions in 2-$dA$.

**1.4.** Let $A$ be any 2-category with respect to $U$. An *adjoint pair* of 1-cells in $A$ is any pair

$$X \underset{g}{\overset{f}{\rightleftarrows}} X'$$

of 1-cells in $A$ for which there are 2-cells $\eta: X \to gf$, $\varepsilon: fg \to X'$ in $A$ with

$$\varepsilon f \cdot f\eta = f \quad \text{and} \quad g\varepsilon \cdot \eta g = g.$$

If $X \underset{g}{\overset{f}{\rightleftarrows}} X'$ is an adjoint pair in $A$, then it is denoted by $\langle f, g, \eta, \varepsilon\rangle$. Let $\boldsymbol{a} = \langle f, g, \eta, \varepsilon\rangle$ be any adjoint pair in $A$. Then $G'\boldsymbol{a} = (f, g)$ may be considered as an object of the category $d_{(1,1)}A$ (respectively, $d_{(3,3)}A$). An *adjunction morphism* (respectively, *inverse adjunction morphism*) from an adjoint pair $\boldsymbol{a} = \langle f, g, \eta, \varepsilon\rangle$ to an adjoint pair $\boldsymbol{a}' = \langle f', g', \eta', \varepsilon'\rangle$ in $A$ is any morphism in $d_{(1,1)}A$ (respectively, $d_{(3,3)}A$) of the form

$$(\langle m, k, \beta\rangle, \langle k, m, \alpha\rangle): G'\boldsymbol{a} \to G'\boldsymbol{a}'$$

such that

$$(1) \qquad k\varepsilon = \varepsilon' k \cdot f'\alpha \cdot \beta g \quad \text{and} \quad \eta' m = g'\beta \cdot \alpha f \cdot m\eta$$

and, respectively,

$$(1^{\leftarrow}) \qquad \varepsilon' k = k\varepsilon \cdot \beta g \cdot f'\alpha \quad \text{and} \quad m\eta = \alpha f \cdot g'\beta \cdot \eta' m.$$

If $(\langle m, k, \beta\rangle, \langle k, m, \alpha\rangle)$ is an adjunction morphism (respectively, inverse adjunction morphism) in $A$, then it may briefly be denoted by $\langle m, k, \beta, \alpha\rangle$. The composition of adjunction morphisms (respectively, inverse adjunction morphisms) in $A$ agrees with the composition of morphisms in $d_{(1,1)}A$ (respectively, $d_{(3,3)}A$). Consider the composition of adjunction morphisms

$$Q = \langle m', k', \beta', \alpha'\rangle\langle m, k, \beta, \alpha\rangle = \langle m'm, k'k, \beta'm \cdot k'\beta, \alpha'k \cdot m'\alpha\rangle.$$

Then by (1) we have

$$\begin{aligned}\varepsilon''k'k \cdot f''(\alpha'k \cdot m'\alpha) \cdot (\beta'm \cdot k'\beta)g &= \varepsilon''k'k \cdot f''\alpha'k \cdot \beta'\alpha \cdot k'\beta g \\ &= k'(\varepsilon'k \cdot f'\alpha \cdot \beta g) = k'k\varepsilon\end{aligned}$$

and

$$\begin{aligned}g''(\beta'm \cdot k'\beta) \cdot (\alpha'k \cdot m'\alpha)f \cdot m'm\eta &= g''\beta'm \cdot \alpha'\beta \cdot m'\alpha f \cdot m'm\eta \\ &= (g''\beta \cdot \alpha'f' \cdot m'\eta')m = \eta''m'm,\end{aligned}$$

and thus $Q$ is an adjunction morphism. The identity $\langle X, X', f, g\rangle$ defined by an adjoint pair $\langle f, g, \eta, \varepsilon\rangle$ is an adjunction morphism. In an analogous way we prove that the composition of inverse adjunction morphisms as well as the identity are inverse adjunction morphisms. Hence the adjoint pairs in $A$, treated as objects, with adjunction morphisms (respectively, inverse adjunction morphisms) considered as morphisms, define the category $\mathrm{Adj}(A)$ (respectively, $\mathrm{Adj}^{\leftarrow}(A)$). Now

$$G'\colon \mathrm{Adj}(A) \to d_{(1,1)} A$$

and, respectively,

$$G'\colon \mathrm{Adj}^{\leftarrow}(A) \to d_{(3,3)} A$$

is a forgetful functor. Moreover, $\mathrm{Adj}(A)$ (respectively, $\mathrm{Adj}^{\leftarrow}(A)$) is the local discrete category of $2\text{-}\mathrm{Adj}(A)$ (respectively, $2\text{-}\mathrm{Adj}^{\leftarrow}(A)$) such that $2\text{-}\mathrm{Adj}(A)(\boldsymbol{a}, \boldsymbol{a}')$ (respectively, $2\text{-}\mathrm{Adj}^{\leftarrow}(A)(\boldsymbol{a}, \boldsymbol{a}')$) is a subcategory of $2\text{-}d_{(1,1)} A(G'\boldsymbol{a}, G'\boldsymbol{a}')$ (respectively, $2\text{-}d_{(3,3)} A(G'\boldsymbol{a}, G'\boldsymbol{a}))$ whose objects are adjunction morphisms (respectively, inverse adjunction morphisms) and morphisms are of the form $(\langle\tau, \sigma\rangle, \langle\sigma, \tau\rangle) = \langle\tau, \sigma\rangle$, where $\boldsymbol{a}$ and $\boldsymbol{a}'$ are any adjoint pairs in $A$. The strong compositions in $2\text{-}\mathrm{Adj}(A)$ (respectively, $2\text{-}\mathrm{Adj}^{\leftarrow}(A)$) agree with the strong compositions in $2\text{-}d_{(1,1)} A$ (respectively, $2\text{-}d_{(3,3)} A$). By 1.2, the morphisms from $\langle m, k, \beta, a\rangle$ to $\langle m', k', \beta', a'\rangle$ in the category $2\text{-}\mathrm{Adj}(A)(\boldsymbol{a}, \boldsymbol{a}')$ (respectively, $2\text{-}\mathrm{Adj}^{\leftarrow}(A)(\boldsymbol{a}, \boldsymbol{a}')$) with $\boldsymbol{a} = \langle f, g, \eta, \varepsilon\rangle$ and $\boldsymbol{a}' = \langle f', g', \eta', \varepsilon'\rangle$ are precisely all pairs $\langle\tau, \sigma\rangle$ of 2-cells $\tau\colon m \to m'$ and $\sigma\colon k \to k'$ in $A$ such that

$$g'\sigma\cdot a = a'\cdot\tau g \quad \text{and} \quad f'\tau\cdot\beta = \beta'\cdot\sigma f \tag{2}$$

and, respectively,

$$\tau g\cdot a = a'\cdot g'\sigma \quad \text{and} \quad \sigma f\cdot\beta = \beta'\cdot f'\tau. \tag{$2^{\leftarrow}$}$$

**Remark.** The adjunction categories $\mathrm{Adj}(A)$ introduced here differ from the categories $\mathrm{sqAdj}(A)$ determined by the adjoint squares in [3] by Gray. In general, there is no forgetful functor from $\mathrm{sqAdj}(A)$ to either $\mathrm{Adj}(A)$ or $\mathrm{Adj}^{\leftarrow}(A)$. In general, there is also no morphism determined by an adjoint square in either $\mathrm{Adj}(A)$ or $\mathrm{Adj}^{\leftarrow}(A)$.

**1.5.** Let $A$ be any 2-category with respect to $U$. The mappings

$$\begin{array}{ccc} \langle f, g, \eta, \varepsilon\rangle & & \langle X, gf, \eta, g\varepsilon f\rangle \\ \Big\downarrow{\scriptstyle\langle m,k,\beta,a\rangle} & \longmapsto & \Big\downarrow{\scriptstyle\langle m, g'\beta\cdot af\rangle\ (\text{respectively}, \langle m, af\cdot g'\beta\rangle)} \\ \langle f', g', \eta', \varepsilon'\rangle & & \langle Y, g'f', \eta', g'\varepsilon' f'\rangle \end{array}$$

define a functor

$$F\colon \mathrm{Adj}(A) \to \mathrm{Mon}(A)$$

and, respectively,

$$F^{\leftarrow}\colon \mathrm{Adj}^{\leftarrow}(A) \to \mathrm{Mon}^{\leftarrow}(A),$$

where

$$X \underset{g}{\overset{f}{\rightleftarrows}} X' \quad \text{and} \quad Y \underset{g'}{\overset{f'}{\rightleftarrows}} Y'$$

are adjoint pairs in $A$.

The mappings

$$\begin{array}{ccc} \boldsymbol{a} = \langle f, g, \eta, \varepsilon\rangle & & \langle F\boldsymbol{a}, g, g\varepsilon\rangle \\ \Big\downarrow{\scriptstyle\langle m,k,\beta,\alpha\rangle} & \mapsto & \Big\downarrow{\scriptstyle(\langle k,m,\alpha\rangle,\langle k,g'\beta\cdot\alpha f\rangle)} \\ \boldsymbol{a}' = \langle f', g', \eta', \varepsilon'\rangle & & \langle F\boldsymbol{a}', g', g'\varepsilon'\rangle \end{array}$$

define a functor

$$F'': \mathrm{Adj}(A) \to \mathrm{Alm}(A),$$

where $\mathrm{Alm}(A)$ is the monad algebra category of $A$ defined in [5]. Moreover, for any adjoint pair $\langle f, g, \eta, \varepsilon\rangle = \boldsymbol{a}$ in $A$, $\langle F\boldsymbol{a}, g\varepsilon\rangle$ is the object of $\mathrm{Kan}(A{\downarrow}A)$, since for any object $\langle S, \xi\rangle$ in the category $2\text{-}dA(X', -)(g, g)$ the pair $\langle X', \xi f\cdot S\eta\rangle$ is the unique morphism of this category from $\langle S, \xi\rangle$ to $\langle F\boldsymbol{a}, g\varepsilon\rangle$, i.e. $\langle F\boldsymbol{a}, g\varepsilon\rangle$ is the right Kan extension of $g$ along $g$, which preserves the 1-cells in $A$.

In general, we do not know any non-trivial functor from $\mathrm{Mon}(A)$ to $\mathrm{Adj}(A)$.

**1.6.** Let $A$ be any 2-category with respect to a universe $U$. Consider the categories $\mathrm{Alm}(A)$ and $\mathrm{Mon}_*(A)$ defined in [5]. By Theorem 2.3 (IIa) in [5] there is an adjunction

$$\Sigma = \langle H, L, \eta^*, \varepsilon^*\rangle: \mathrm{Mon}_*(A) \rightharpoonup \mathrm{Alm}(A).$$

Let us denote by $\mathrm{Kr}(A)$ the full subcategory of $\mathrm{Alm}(A)$ defined by all objects of the form $Hb$, where $b$ is any object in $\mathrm{Mon}_*(A)$. Let $\varkappa = \varkappa^{(j)}$ $(j = 1, 2, \ldots, 5)$ be any admissible sequence with respect to $A$ and $(1, 0)$ given in 1.5.1 of [5]. By $\mathrm{Kr}(A)(\varkappa)$ we denote the subcategory of $\mathrm{Kr}(A)$ defined by all objects and all morphisms in $\mathrm{Alm}(A)(\varkappa)$ (see [5]). The pairs

$$\mathrm{kr} = \langle \mathrm{Mon}_*(A), \mathrm{Kr}(A)\rangle \quad\text{and}\quad \mathrm{kr}_\varkappa = \langle \mathrm{Mon}_*(A)(\varkappa), \mathrm{Kr}(A)(\varkappa)\rangle,$$

where $\mathrm{Mon}_*(A)(\varkappa)$ is defined as in [5], fulfil conditions $(S_1)$-$(S_4)$ of Definition 2.2 in [5] and thus, by Theorem 2.3 (I) of that paper, we have the adjunctions

$$\Sigma_{\mathrm{kr}} = \langle H_{\mathrm{kr}}, L_{\mathrm{kr}}, \eta_{\mathrm{kr}}, \varepsilon_{\mathrm{kr}}\rangle: \mathrm{Mon}_*(A) \rightharpoonup \mathrm{Kr}(A)$$

and

$$\Sigma_{\mathrm{kr}\varkappa} = \langle H_{\mathrm{kr}\varkappa}, L_{\mathrm{kr}\varkappa}, \eta_{\mathrm{kr}\varkappa}, \varepsilon_{\mathrm{kr}\varkappa}\rangle: \mathrm{Mon}_*(A)(\varkappa) \rightharpoonup \mathrm{Kr}(A)(\varkappa).$$

The category $\mathrm{Mon}_*(A)(\varkappa)$ is isomorphic to $A(X, X')$ if $\varkappa = (X, \boldsymbol{T})$, where $X' = \mathrm{dom}T$, and to the category $dA(-, X')$ if $\varkappa = (-, \boldsymbol{T})$, where $X' = \mathrm{dom}T$. If $A = 2\text{-CAT}(U)$, then $\Sigma_{\mathrm{kr}(1,\boldsymbol{T})}$ is the usual Kleisli adjunction defined by a monad $\boldsymbol{T}$ of a category $X'$ (see [4]). The monads of the above-given adjunctions are denoted by $D_{\mathrm{kr}}$ and $D_{\mathrm{kr}\varkappa}$, respectively.

Let $\boldsymbol{s} = \langle m, \psi\rangle$ be any fixed monad morphism in $A$ from $\boldsymbol{T} = \langle X', T, \eta, \mu\rangle$ to $\boldsymbol{T}' = \langle Y', T', \eta', \mu'\rangle$. With any object $\boldsymbol{a} = \langle \boldsymbol{T}, Tf, \mu f\rangle$ in

$\mathrm{Kr}(A)(-, T)$ we associate an object $\mathrm{kr}(s)(a) = \langle T', T'mf, \mu'mf\rangle$ in $\mathrm{Kr}(A)(-, T')$. To any morphism

$$Q = (\langle n, X', \varphi\rangle, \langle n, T\rangle): a \to a' = \langle T, Tf', \mu f'\rangle$$

in $\mathrm{Kr}(A)(-, T)$ there corresponds a morphism $Q_1$ in $dA(-, Y')$ of the form

$$\langle n, Y', q\rangle: mf \to T'mf', \quad \text{where } q = \psi f'n\cdot m\varphi\cdot m\eta f,$$

and thus, by the adjunction $\Sigma_{\mathrm{kr}(-,T)}$, there is a unique morphism $Q_1^{\neq}$ from $\mathrm{kr}(s)(a)$ to $\mathrm{kr}(s)(a')$, being the free extension of $Q_1$. We have

$$\begin{aligned} LQ_1^{\neq} = L\varepsilon_{a'}^{*}\cdot D_{\mathrm{kr}(-,T)}Q_1 &= \langle X, Y', \mu'mf'\rangle\langle n, Y', T'q\rangle \\ &= \langle n, Y', \mu'mf'n\cdot T'q\rangle = \langle n, Y', s(Q)\rangle, \end{aligned}$$

where

$$s(Q) = \mu'mf'n\cdot T'\psi f'n\cdot T'm\varphi\cdot T'm\eta f. \tag{3}$$

The morphism $Q_1^{\neq}$ is denoted by $\mathrm{kr}(s)(Q)$. Thus we have defined the mappings

$$\mathrm{kr}(s): \mathrm{Kr}(A)(-, T) \to \mathrm{Kr}(A)(-, T')$$

on objects and on morphisms. We prove that $\mathrm{kr}(s)$ preserves the composition. First, let us observe that for any above-given morphism $Q: a \to a'$ there is exactly one morphism

$$Q' = (\langle n, Y', \gamma\rangle, \langle n, T'\rangle): \mathrm{kr}(s)(a) \to \mathrm{kr}(s)(a')$$

such that

$$Q'h_a = h_{a'}Q, \tag{4}$$

where

$$h_a = (\langle X, m, \psi f\rangle, \langle X, \psi\rangle): a \to \mathrm{kr}(s)(a)$$

(it is a monad algebra morphism in $A$). Indeed, if $Q'$ and $Q_1'$ fulfil (4), then

$$\begin{aligned} \eta^{*}_{\langle mf, T'\rangle}LQ' &= \langle n, Y', \gamma\cdot\eta'mf\rangle = \langle n, Y', \gamma\cdot\psi f\cdot m\eta f\rangle \\ &= \langle n, Y', \gamma_1\cdot\psi f\cdot m\eta f\rangle = \eta^{*}_{\langle mf, T'\rangle}LQ_1', \end{aligned}$$

and thus $Q' = Q_1'$. Moreover,

$$s(Q)\cdot\psi f = \psi f'n\cdot m\varphi, \tag{5}$$

since

$$\begin{aligned} s(Q)\cdot\psi f = \mu'mf'n\cdot T'\psi f'n\cdot T'm\varphi\cdot\psi\eta f &= \mu'mf'n\cdot T'\psi f'n\cdot\psi\varphi\cdot mT\eta f \\ = \psi f'n\cdot m(\varphi\cdot\mu f)\cdot mT\eta f &= \psi f'n\cdot m\varphi, \end{aligned}$$

and thus $Q' = \mathrm{kr}(s)(Q)$ is the unique morphism which fulfils (4). Hence $\mathrm{kr}(s)$ preserves the composition, $\mathrm{kr}(s)$ is the functor from $\mathrm{Kr}(A)(-, T)$ to $\mathrm{Kr}(A)(-, T')$ and

$$s(Q_1Q) = s(Q_1)n\cdot s(Q) \tag{6}$$

holds. Moreover,

$$\mathrm{kr}(\boldsymbol{s})H_{\mathrm{kr}(-,\boldsymbol{T})} = H_{\mathrm{kr}(-,\boldsymbol{T}')}\tilde{m},$$

where $\tilde{m}$ is the functor from $dA(-, X')$ to $dA(-, Y')$ satisfying $\tilde{m}f = mf$ on objects, and $\tilde{m}\langle n, X', \varphi\rangle = \langle n, Y', m\varphi\rangle$ on morphisms. Indeed,

$$\begin{aligned}
\mathrm{kr}(\boldsymbol{s})H_{\mathrm{kr}(-,\boldsymbol{T})}\langle n, X', \gamma\rangle &= \mathrm{kr}(\boldsymbol{s})(\langle n, X', T\gamma\rangle,\ \langle n, T\rangle) \\
&= (\langle n,\ Y', \mu' mf'n\cdot T'\psi f'n\cdot T'mT\gamma\cdot T'm\eta f\rangle,\ \langle n, T'\rangle) \\
&= (\langle n,\ Y', \mu' mf'n\cdot T'\psi f'n\cdot T'm\eta\gamma\rangle,\ \langle n, T'\rangle) \\
&= (\langle n,\ Y', \mu' mf'n\cdot T'\psi f'n\cdot T'm\eta f'n\cdot T'm\gamma\rangle,\ \langle n, T'\rangle) \\
&= (\langle n,\ Y', \mu' mf'n\cdot T'\eta' mf'n\cdot T'm\gamma\rangle,\ \langle n, T'\rangle) \\
&= (\langle n,\ Y', T'm\gamma\rangle, \langle n, T'\rangle) = H_{\mathrm{kr}(-,\boldsymbol{T}')}\langle n,\ Y',\ m\gamma\rangle \\
&= H_{\mathrm{kr}(-,\boldsymbol{T}')}\tilde{m}\langle n, X', \gamma\rangle.
\end{aligned}$$

Let us observe that the $\mathrm{kr}(\boldsymbol{s})$-images of both objects and morphisms of $\mathrm{Kr}(A)(X, \boldsymbol{T})$ belong to the category $\mathrm{Kr}(A)(X, \boldsymbol{T}')$. Thus, for any 0-cell $X$ in $A$, $\mathrm{kr}(\boldsymbol{s})$ may be regarded as a functor from $\mathrm{Kr}(A)(X, \boldsymbol{T})$ to $\mathrm{Kr}(A)(X, \boldsymbol{T}')$ such that

$$\mathrm{kr}(s)H_{\mathrm{kr}(X,\boldsymbol{T})} = H_{(\mathrm{kr}X,\boldsymbol{T}')}\tilde{m},$$

where $\tilde{m}\colon A(X, X') \to A(X, Y')$ is the restriction of the above functor $\tilde{m}$.

Let $A$ be any 2-category with respect to $U$. Then we define $A'$ as $2\text{-}\mathrm{Cat}(U)$ if all categories in $A$ (i.e. $A_0$ and $A(X, Y)$ with $X, Y$ in $A_0$) are objects in $\mathrm{Cat}(U)$, and as $2\text{-}\mathrm{CAT}(U)$ in the opposite case. Hence

(I) The mappings

$$\begin{array}{ccc}
\boldsymbol{T} & & \Sigma_{\mathrm{kr}(-,\boldsymbol{T})} \\
\Big\downarrow{\scriptstyle s=\langle m,\psi\rangle} & \longmapsto & \Big\downarrow{\scriptstyle \langle\tilde{m},\mathrm{kr}(s),\mathrm{id},\psi^*\rangle} \\
\boldsymbol{T}' & & \Sigma_{\mathrm{kr}(-,\boldsymbol{T}')}
\end{array}$$

define a functor

$$K\colon \mathrm{Mon}(A) \to \mathrm{Adj}(A'),$$

where

$$\psi^*\langle T, Tf, \mu f\rangle = \langle X, Y', \psi f\rangle\colon mTf \to T'mf.$$

(II) For each 0-cell $X$ in $A$ the mappings

$$\begin{array}{ccc}
\boldsymbol{T} & & \Sigma_{\mathrm{kr}(X,\boldsymbol{T})} \\
\Big\downarrow{\scriptstyle s=\langle m,\psi\rangle} & \longmapsto & \Big\downarrow{\scriptstyle \langle\tilde{m},\mathrm{kr}(s),\mathrm{id},\psi^*\rangle} \\
\boldsymbol{T}' & & \Sigma_{\mathrm{kr}(X,\boldsymbol{T}')}
\end{array}$$

define a functor

$$K_X\colon \mathrm{Mon}(A) \to \mathrm{Adj}(A').$$

Now consider the adjunctions

$$\Sigma_{(-,\boldsymbol{T})} = \langle H_{(-,\boldsymbol{T})}, L_{(-,\boldsymbol{T})}, \eta_{(-,\boldsymbol{T})}, \varepsilon_{(-,\boldsymbol{T})}\rangle\colon dA(-, \boldsymbol{T}) \rightharpoonup \mathrm{Alm}(A)(-, \boldsymbol{T})$$

and

$$\Sigma_{(X,T)} = \langle H_{(X,T)}, L_{(X,T)}, \eta_{(X,T)}, \varepsilon_{(X,T)}\rangle : A(X, X') \to \mathrm{Alm}(A)(X, \boldsymbol{T}),$$

where $\boldsymbol{T} = \langle X', T, \eta, \mu\rangle$ is any monad in $A$ and $X$ is any 0-cell in $A$, as in Theorem 2.3 (IIb) of [5] for $\varkappa = \varkappa^{(j)}$ with $j = 5, 4$. Let $\boldsymbol{s} = \langle m, \psi\rangle$ be any inverse monad morphism in $A$ from a monad $\boldsymbol{T} = \langle X', T, \eta, \mu\rangle$ to a monad $\boldsymbol{T}' = \langle Y', T', \eta', \mu'\rangle$. Then the formulas

(7) $\quad \mathrm{al}(\boldsymbol{s})(\boldsymbol{a}) = \langle T', mf, ma\cdot\psi f\rangle, \quad \mathrm{al}(\boldsymbol{s})(Q) = (\langle n, Y', m\varphi\rangle, \langle n, T'\rangle),$

where

$$\boldsymbol{a} = \langle T, f, a\rangle \quad \text{and} \quad Q = (\langle n, X', \varphi\rangle, \langle n, T\rangle),$$

define a functor

$$\mathrm{al}(\boldsymbol{s}) : \mathrm{Alm}(A)(-, \boldsymbol{T}) \to \mathrm{Alm}(A)(-, \boldsymbol{T}').$$

For any 0-cell $X$ in $A$ the al(s)-images of the objects and morphisms of $\mathrm{Alm}(A)(X, \boldsymbol{T})$ belong to $\mathrm{Alm}(A)(X, \boldsymbol{T}')$, so al(s) may be considered as the functor from $\mathrm{Alm}(A)(X, \boldsymbol{T})$ to $\mathrm{Alm}(A)(X, \boldsymbol{T}')$. Let $A$ and $A'$ have the same meaning as before. Hence

($\mathrm{I}^{\leftarrow}$) The mappings

$$\begin{array}{ccc} \boldsymbol{T} & & \Sigma_{(-,T)} \\ \Big\downarrow{\scriptstyle s=\langle m,\psi\rangle} & \longmapsto & \Big\downarrow{\scriptstyle \langle \tilde{m}, \mathrm{al}(s), \psi_*, \mathrm{id}\rangle} \\ \boldsymbol{T}' & & \Sigma_{(-,T')} \end{array}$$

define a functor

$$K^{\leftarrow} : \mathrm{Mon}^{\leftarrow}(A) \to \mathrm{Adj}^{\leftarrow}(A'),$$

where

$$\psi_*(f) = (\langle X, Y', \psi f\rangle, \langle X, T'\rangle).$$

($\mathrm{II}^{\leftarrow}$) For each 0-cell $X$ in $A$ the mappings

$$\begin{array}{ccc} \boldsymbol{T} & & \Sigma_{(X,T)} \\ \Big\downarrow{\scriptstyle s=\langle m,\psi\rangle} & \longmapsto & \Big\downarrow{\scriptstyle \langle \tilde{m}, \mathrm{al}(s), \psi_*, \mathrm{id}\rangle} \\ \boldsymbol{T}' & & \Sigma_{(X,T')} \end{array}$$

define a functor

$$K_X^{\leftarrow} : \mathrm{Mon}^{\leftarrow}(A) \to \mathrm{Adj}^{\leftarrow}(A').$$

Obviously, $L_{(-,T)}$ and $L_{(X,T)}$ are the objects of the category $\mathrm{Kan}(A{\downarrow}A)$. Since $\mathrm{pr}_4 K^{\leftarrow}(\boldsymbol{s}) = \mathrm{id}$, the mappings

$$\begin{array}{ccc} \boldsymbol{T} & & L_{(-,T)}\ (\text{or } L_{(X,T)}) \\ \Big\downarrow{\scriptstyle s=\langle m,\psi\rangle} & \longmapsto & \Big\downarrow{\scriptstyle \langle \mathrm{al}(s), \tilde{m}\rangle} \\ \boldsymbol{T}' & & L_{(-,T')}\ (\text{or } L_{(X,T')}) \end{array}$$

define the functors

$$\mathrm{Sem}(-, A) : \mathrm{Mon}^{\leftarrow}(A) \to \mathrm{Kan}(A'{\downarrow}A')$$

and

$$\mathrm{Sem}(X, A)\colon \mathrm{Mon}^{\leftarrow}(A) \to \mathrm{Kan}(A'{\downarrow}A'),$$

where $X$ is any 0-cell in $A$.

Let us assume that $A = 2\text{-}\mathrm{CAT}(U)$. Then $A' = A$ and we have the functors

$$F\colon \mathrm{Adj}(A) \to \mathrm{Mon}(A), \quad K\colon \mathrm{Mon}(A) \to \mathrm{Adj}(A),$$
$$K_X\colon \mathrm{Mon}(A) \to \mathrm{Adj}(A)$$

and

$$F^{\leftarrow}\colon \mathrm{Adj}^{\leftarrow}(A) \to \mathrm{Mon}^{\leftarrow}(A), \quad K^{\leftarrow}\colon \mathrm{Mon}^{\leftarrow}(A) \to \mathrm{Adj}^{\leftarrow}(A),$$
$$K_X^{\leftarrow}\colon \mathrm{Mon}^{\leftarrow}(A) \to \mathrm{Adj}^{\leftarrow}(A).$$

We do not know any adjunctions between those functors.

**2. Some adjunctions between the categories in adjunction and monad 2-categories.** Let $A = 2\text{-}\mathrm{CAT}(U)$, where $U$ is any universe.

THEOREM 2.1. *For any* 0*-cell $X$ in $A$, for any monad $T = \langle X', T, \eta, \mu\rangle$ in $A$ and for any adjunction*

$$\Sigma = \langle f, g, \eta', \varepsilon'\rangle\colon Y \rightharpoonup Y'$$

*in $A$, there is an adjunction*

$$B_X = \langle a_X, b_X, \eta_X^0, \varepsilon_X^0\rangle\colon 2\text{-}\mathrm{Adj}(A)\big(K_X(T), \Sigma\big) \rightharpoonup 2\text{-}\mathrm{Mon}(A)\big(D_{\mathrm{kr}(X,T)}, F(\Sigma)\big)$$

*such that the functor $b_X$ is full and faithful.*

Proof. Let

$$C = 2\text{-}\mathrm{Adj}(A)\big(K_X(T), \Sigma\big) \quad \text{and} \quad C' = 2\text{-}\mathrm{Mon}(A)\big(D_{\mathrm{kr}(X,T)}, F(\Sigma)\big).$$

Let $r = \langle m, k, \beta, a\rangle$ be any object in $C$, i.e. any diagram

$$\begin{tikzcd} A(X, X') \arrow[r, "H_{\mathrm{kr}}"] \arrow[d, "m"'] \arrow[dr, phantom, "\beta\Downarrow"] & \mathrm{Kr}(A)(X, T) \arrow[r, "L_{\mathrm{kr}}"] \arrow[d, "k"] \arrow[dr, phantom, "a\Downarrow"] & A(X, X') \arrow[d, "m"] \\ Y \arrow[r, "f"'] & Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

in $A$ which is an adjunction morphism. We write $L_{\mathrm{kr}}$, $H_{\mathrm{kr}}$, and $D_{\mathrm{kr}}$ instead of $L_{\mathrm{kr}(X,T)}$, $H_{\mathrm{kr}(X,T)}$, and $D_{\mathrm{kr}(X,T)}$, respectively. We put

$$s = a_X(r) = \langle m, \psi\rangle, \quad \text{where } \psi = g\beta\cdot aH_{\mathrm{kr}}.$$

The pair $s$ is an object of $C'$. Indeed, by (1) for $r$ we have

$$\psi\cdot m\eta_{\mathrm{kr}} = g\beta\cdot aH_{\mathrm{kr}}\cdot m\eta_{\mathrm{kr}} = m\eta'$$

and, moreover,

$$\begin{aligned} g\varepsilon' fm\cdot gf(g\beta\cdot aH_{\mathrm{kr}})\cdot(g\beta\cdot aH_{\mathrm{kr}})D_{\mathrm{kr}} &= g\beta\varepsilon'\cdot gfaH_{\mathrm{kr}}\cdot g\beta L_{\mathrm{kr}}H_{\mathrm{kr}}\cdot aH_{\mathrm{kr}}D_{\mathrm{kr}} \\ &= g\beta\cdot gk\varepsilon_{\mathrm{kr}}H_{\mathrm{kr}}\cdot aH_{\mathrm{kr}}D_{\mathrm{kr}} = g\beta\cdot a\varepsilon_{\mathrm{kr}}H_{\mathrm{kr}} = g\beta\cdot(a\cdot mL_{\mathrm{kr}}\varepsilon_{\mathrm{kr}})H_{\mathrm{kr}} \\ &= \psi\cdot mL_{\mathrm{kr}}\varepsilon_{\mathrm{kr}}H_{\mathrm{kr}} = \psi\cdot m\mu_{\mathrm{kr}}, \end{aligned}$$

which implies that $s$ is an object of $C'$. Let $\langle \tau, \sigma \rangle$ be any morphism in $C$ from $r = \langle m, k, \beta, a \rangle$ to $r' = \langle m', k', \beta', a' \rangle$. Then, by putting $a_X(\langle \tau, \sigma \rangle) = \tau$ we obtain a morphism in $C'$ from $s = a_X(r) = \langle m, \psi \rangle$ to $s' = a_X(r') = \langle m', \psi' \rangle$. Indeed, by (2) we have

$$gf\tau\cdot\psi = gf\tau\cdot g\beta\cdot aH_{\mathrm{kr}} = g(f\tau\cdot\beta)\cdot aH_{\mathrm{kr}} = g(\beta'\cdot\sigma H_{\mathrm{kr}})\cdot aH_{\mathrm{kr}}$$
$$= g\beta'\cdot(g\sigma\cdot a)H_{\mathrm{kr}} = g\beta'\cdot a'H_{\mathrm{kr}}\cdot\tau L_{\mathrm{kr}}H_{\mathrm{kr}} = \psi'\cdot\tau L_{\mathrm{kr}}H_{\mathrm{kr}}.$$

Obviously, the mapping $a_X\colon C \to C'$ preserves the composition, so it is a functor. Now we define a functor $b_X\colon C' \to C$. For each object $s = \langle m, \psi \rangle$ in $C'$, i.e. for each monad morphism $s$ from $D_{\mathrm{kr}}$ to $F(\Sigma) = T' = \langle Y, gf, \eta', g\varepsilon' f \rangle$, we put $b_X(s) = \langle m, \overline{m}, \beta, a \rangle$, where $\beta = \mathrm{id} = fm$ and $a = \overline{\psi}$, i.e. $a_{\langle T, Tf', \mu f' \rangle} = \psi f'$ and $\overline{m}$ is defined by the formulas

$$(8) \qquad \overline{m}(\langle T, Tf', \mu f' \rangle) = fmf', \qquad \overline{m}(Q) = \varepsilon' fmf''\cdot f\psi f''\cdot fm\varphi\cdot fm\eta_{\mathrm{kr}} f',$$

where $\langle T, Tf', \mu f' \rangle$ is any object of $\mathrm{Kr}(A)(X, T)$ and

$$Q = (\langle X, X', \varphi \rangle, \langle X, T \rangle)\colon \langle T, Tf', \mu f' \rangle \to \langle T, Tf'', \mu f'' \rangle$$

is any morphism of $\mathrm{Kr}(A)(X, T)$.

Using (5) for $f = f'$, $f' = f''$ and $n = X$ we have

$$(9) \qquad \overline{m}(Q) = \varepsilon' fmf''\cdot fs(Q)\cdot f\eta' mf',$$

where $s(Q)$ is given by (3) for $n = X$, $f = f'$ and $f' = f''$. Moreover, using (5) and (3) we obtain

$$\overline{m}(Q_1\cdot Q) = \varepsilon' fmf'''\cdot fs(Q_1)\cdot f\psi f'''\cdot fm\varphi\cdot fm\eta f'$$
$$= (\varepsilon' f\varepsilon') mf'''\cdot fgf\psi f'''\cdot fgfm\varphi_1\cdot fgfm\eta f''\cdot f\psi f''\cdot fm\varphi\cdot fm\eta f'$$
$$= \varepsilon' fmf'''\cdot\varepsilon' f\psi\cdot fgfm\varphi_1\cdot fgfm\eta f''\cdot f\psi f''\cdot fm\varphi\cdot fm\eta f'$$
$$= \varepsilon' fmf'''\cdot f\psi f'''\cdot\varepsilon' fm\varphi_1\cdot fgfm\eta f''\cdot f\psi f''\cdot fm\varphi\cdot fm\eta f'$$
$$= \varepsilon' fmf'''\cdot f\psi f'''\cdot fm\varphi_1\cdot(\varepsilon' fm\eta) f''\cdot f\psi f''\cdot fm\varphi\cdot fm\eta f'$$
$$= \overline{m}(Q_1)\cdot\overline{m}(Q),$$

which implies that $\overline{m}$ preserves the composition, and thus $\overline{m}$ is a functor. By (9) and (5),

$$\overline{m}H_{\mathrm{kr}}Q = \varepsilon' fmf''\cdot fs(H_{\mathrm{kr}}Q)\cdot\psi f'\cdot m\eta_{\mathrm{kr}} f' = \varepsilon' fmf''\cdot f(\psi\varphi\cdot m\eta_{\mathrm{kr}} f')$$
$$= \varepsilon' fm\varphi\cdot f\eta' mf' = fm\varphi\cdot\varepsilon' fmf'\cdot f\eta' mf' = fm\varphi$$

and, therefore,

$$\overline{m}H_{\mathrm{kr}} = fm.$$

By (5), (3) and (8), $\overline{\psi}$ is a natural transformation. Since

$$\overline{m}\varepsilon_{\mathrm{kr}}\langle T, Tf', \mu f' \rangle = \overline{m}(\langle X, X', \mu f' \rangle, \langle X, T \rangle) = \varepsilon' fmf'\cdot f\mu' mf'\cdot f(\eta'\psi) f'$$
$$= \varepsilon' fmf'\cdot f\psi f' = \varepsilon'\overline{m}\langle T, Tf', \mu f' \rangle\cdot fa\langle T, Tf', \mu f' \rangle,$$

i.e.

$$\overline{m}\varepsilon_{\mathrm{kr}} = \varepsilon'\overline{m}\cdot fa,$$

and

$$\eta' mf' = \psi f'\cdot m\eta_{\mathrm{kr}}f' = aH_{\mathrm{kr}}f'\cdot m\eta_{\mathrm{kr}}f', \quad \text{i.e.} \quad \eta' m = aH_{\mathrm{kr}}\cdot m\eta_{\mathrm{kr}},$$

$b_X(s)$ is an object of $C$ by (1). For any morphism

$$\tau\colon s = \langle m, \psi\rangle \to s' = \langle m', \psi'\rangle$$

in $C'$, by putting

$$b_X(\tau) = \langle\tau, \overline{f\tau}\rangle, \quad \text{where } \overline{f\tau}(\langle T, Tf', \mu f'\rangle) = f\tau f',$$

we obtain a morphism in $C$ from $b_X(s)$ to $b_X(s')$. Indeed, the first equality of (2) holds by assumption on $\tau$. The second equality of (2) is obvious by the definition of $b_X$. From the definition it follows easily that $b_X$ preserves the composition. Thus $b_X\colon C' \to C$ is a functor for which $a_X b_X = 1_{C'}$. We now define the unit and counit of an adjunction between the functors $a_X$ and $b_X$. We put $\varepsilon^0 = \mathrm{id}$ and

$$\eta^0\langle m, k, \beta, a\rangle = \langle m, \overline{\beta}\rangle\colon \langle m, k, \beta, a\rangle \to b_X a_X\langle m, k, \beta, a\rangle$$
$$= b_X\langle m, g\beta\cdot aH_{\mathrm{kr}}\rangle = \langle m, \overline{m}, \mathrm{id}, \overline{g\beta\cdot aH_{\mathrm{kr}}}\rangle$$

and we can see by (2) that $\langle m, \overline{\beta}\rangle$ is a morphism in $C$. This morphism is natural in $\langle m, k, \beta, a\rangle$, since for any morphism $\langle\tau, \sigma\rangle$ in $C$ we have

$$b_X a_X\langle\tau, \sigma\rangle\cdot\langle m, \overline{\beta}\rangle = \langle\tau, \overline{f\tau}\rangle\cdot\langle m, \overline{\beta}\rangle = \langle\tau, \overline{f\tau}\cdot\overline{\beta}\rangle = \langle\tau, \overline{f\tau\cdot\beta}\rangle$$
$$= \langle\tau, \overline{\beta'\cdot\sigma H_{\mathrm{kr}}}\rangle = \langle\tau, \overline{\beta'}\cdot\sigma\rangle = \langle m', \overline{\beta'}\rangle\cdot\langle\tau, \sigma\rangle.$$

Moreover,

$$\varepsilon^0 a_X\langle m, k, \beta, a\rangle\cdot a_X\eta^0\langle m, k, \beta, a\rangle = \varepsilon^0\langle m, g\beta\cdot aH_{\mathrm{kr}}\rangle\cdot a_X\langle m, \overline{\beta}\rangle$$
$$= m\cdot m = m = a_X\langle m, k, \beta, a\rangle$$

and

$$b_X\varepsilon^0\langle m, \psi\rangle\cdot\eta^0 b_X\langle m, \psi\rangle = \langle m, \overline{fm}\rangle\cdot\eta^0\langle m, \overline{m}, \mathrm{id}, \overline{\psi}\rangle$$
$$= \langle m, \overline{m}\rangle\cdot\langle m, \mathrm{id}\rangle = \langle m, \overline{m}\rangle = b_X\langle m, \psi\rangle,$$

i.e.

$$\varepsilon^0 a_X\cdot a_X\eta^0 = a_X \quad \text{and} \quad b_X\varepsilon^0\cdot\eta^0 b_X = b_X.$$

Thus $B_X = \langle a_X, b_X, \eta^0, \varepsilon^0\rangle$ is an adjunction. Since the counit of this adjunction is identity, $b_X$ is full and faithful.

THEOREM 2.2. *For any 0-cell $X$ in $A$, for any monad $T = \langle X', T, \eta, \mu\rangle$ in $A$ and for any adjunction*

$$\Sigma = \langle f, g, \eta', \varepsilon'\rangle\colon Y \rightharpoonup Y'$$

*in $A$, there is an adjunction*

$$B_X^{\leftarrow} = \langle a_X^{\leftarrow}, b_X^{\leftarrow}, \eta^{0\leftarrow}, \varepsilon^{0\leftarrow}\rangle\colon \text{2-Adj}^{\leftarrow}(A)\big(\Sigma, K_X^{\leftarrow}(T)\big)$$
$$\rightharpoonup \text{2-Mon}^{\leftarrow}(A)\big(F^{\leftarrow}(\Sigma), D_{(X,T)}\big)$$

*such that the functor $b_X^{\leftarrow}$ is full and faithful, where $D_{(X,T)}$ is the monad defined by the adjunction $\Sigma_{(X,T)}$.*

Proof. Let $r = \langle m, k, \beta, a\rangle$ be any object of the category

$$C = 2\text{-Adj}^{\leftarrow}(A)(\Sigma, K_X^{\leftarrow}(T)),$$

i.e. any diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "m"'] & Y' \arrow[r, "g"] \arrow[d, "k"] \arrow[dl, phantom, "\beta\uparrow"] & Y \arrow[d, "m"] \arrow[dl, phantom, "a\uparrow"] \\ A(X, X') \arrow[r, "H"'] & \mathrm{Alm}(A)(X, T) \arrow[r, "L"'] & A(X, X') \end{tikzcd}$$

which is an inverse adjunction morphism, where $H$ is $H_{(X,T)}$ and $L$ is $L_{(X,T)}$. Putting

$$s = a_X^{\leftarrow}(r) = \langle m, af\cdot L\beta\rangle$$

we obtain an inverse monad morphism. By ($1^{\leftarrow}$) we have

$$\psi\cdot\eta^* m = af\cdot L\beta\cdot\eta^* m = m\eta'$$

and

$$\begin{aligned} m\mu'\cdot\psi T'\cdot D_{(X,T)}\psi &= mg\varepsilon' f\cdot(af\cdot L\beta)gf\cdot LH(af\cdot L\beta) \\ &= a\varepsilon' f\cdot L\beta gf\cdot LHaf\cdot LHL\beta = af\cdot L(k\varepsilon'\cdot\beta g\cdot Ha)f\cdot LHL\beta \\ &= af\cdot L\varepsilon^*\beta = af\cdot L\beta\cdot L\varepsilon^* Hm = \psi\cdot\mu^* m, \end{aligned}$$

which implies that $s$ is an object of the category

$$C' = 2\text{-Mon}^{\leftarrow}(A)(F^{\leftarrow}(\Sigma), D_{(X,T)}).$$

Let

$$\langle\tau, \sigma\rangle : r = \langle m, k, \beta, a\rangle \to r' = \langle m', k', \beta', a'\rangle$$

be any morphism in $C$ and let

$$s = a_X^{\leftarrow}(r) = \langle m, \psi\rangle, \qquad s' = a_X^{\leftarrow}(r') = \langle m', \psi'\rangle.$$

We put $\tau = a_X^{\leftarrow}(\langle\tau, \sigma\rangle)$, and thus we obtain a morphism in $C'$ from $s$ to $s'$, since by ($2^{\leftarrow}$) we have

$$\begin{aligned} \tau gf\cdot\psi &= (\tau g\cdot a)f\cdot L\beta = (a'\cdot L\sigma)f\cdot L\beta = a'f\cdot L(\sigma f\cdot\beta) \\ &= a'f\cdot L(\beta'\cdot H\tau) = \psi'\cdot LH\tau. \end{aligned}$$

Obviously, the mapping $a_X^{\leftarrow}: C \to C'$ preserves the composition, and thus it is a functor.

Now we define a functor $b_X^{\leftarrow}: C' \to C$. For any object $s = \langle m, \psi\rangle$ in $C'$, i.e. for any diagram

$$\begin{tikzcd} Y \arrow[r, "gf"] \arrow[d, "m"'] & Y \arrow[d, "m"] \arrow[dl, phantom, "\psi\uparrow"] \\ A(X, X') \arrow[r, "LH"'] & A(X, X') \end{tikzcd}$$

which is an inverse monad morphism in $A$, we put

$$r = b_X^{\leftarrow}(s) = (m, \overline{m}, \beta, a\rangle,$$

where

(10) $$\overline{m}\colon Y' \to \mathrm{Alm}(A)(X, T)$$

is a mapping on objects and on morphisms such that

$$\overline{m}(y_0') = \langle T, mgy_0', mg\varepsilon' y_0' \cdot \psi g y_0' \rangle$$

for any object $y_0'$ in $Y'$, and

$$\overline{m}(Q) = (\langle X, X', mgQ\rangle, \langle X, T\rangle)$$

for any morphism $Q\colon y_0' \to y_0''$ in $Y'$,

(11) $$a = \mathrm{id} = mg = L\overline{m},$$

(12) $$\beta = \overline{\psi} \quad \text{with } \overline{\psi} y_0 = (\langle X, X', \psi y_0\rangle, \langle X, T\rangle)$$

for each object $y_0$ in $Y$.

Thus we obtain an object in $C$. Indeed, $\overline{m}(y_0')$ is a monad algebra, since

$$mg\varepsilon' y_0' \cdot \psi g y_0' \cdot \eta m g y_0' = m(g\varepsilon' \cdot \eta' g) y_0' = mgy_0'$$

and

$$mg\varepsilon' y_0' \cdot \psi g y_0' \cdot T(mg\varepsilon' y_0' \cdot \psi g y_0') = mg\varepsilon' y_0' \cdot \psi g y_0' \cdot LHmg\varepsilon' y_0' \cdot LH\psi g y_0'$$
$$= mg\varepsilon' y_0' \cdot (\psi g \varepsilon') y_0' \cdot LH\psi g y_0' = mg\varepsilon'\varepsilon' y_0' \cdot \psi g f g y_0' \cdot LH\psi g y_0'$$
$$= mg\varepsilon' y_0' \cdot (m\mu' \cdot \psi g f \cdot LH\psi) g y_0' = mg\varepsilon' y_0' \cdot (\psi \cdot \mu^* m) g y_0' = mg\varepsilon' y_0' \cdot \psi g y_0' \cdot \mu m g y_0'.$$

$\overline{m}(Q)$ is a monad algebra morphism, since

$$mgQ \cdot mg\varepsilon' y_0' \cdot \psi g y_0' = mg(\varepsilon' Q) \cdot \psi g y_0' = mg\varepsilon' y_0'' \cdot \psi g Q$$
$$= mg\varepsilon' y_0'' \cdot \psi g y_0'' \cdot LHmgQ = mg\varepsilon' y_0'' \cdot \psi g y_0'' \cdot TmgQ.$$

Hence, by $mg(Q_1) \cdot mg(Q) = mg(Q_1 \cdot Q)$, the mapping $\overline{m}$ defined in (10) is a functor. Obviously, $L\overline{m} = mg$. The mapping $\overline{\psi}\colon Hm \to \overline{m}f$ defined in (12) is a natural transformation since $\overline{\psi} y_0$ is an algebra morphism for any object $y_0$ in $Y$. Indeed,

$$\psi y_0 \cdot \mu m y_0 = \psi y_0 \cdot \mu^* m y_0 = m\mu' y_0 \cdot \psi g f y_0 \cdot LH\psi y_0 = mg\varepsilon' f y_0 \cdot \psi g f y_0 \cdot T\psi y_0.$$

For any morphism $q\colon y_0 \to y_1$ in $Y$ we obtain

$$\overline{m} f q \cdot \overline{\psi} y_0 = (\langle X, X', \psi q\rangle, \langle X, T\rangle) = (\langle X, X', \psi y_1 \cdot LHmq\rangle, \langle X, T\rangle)$$
$$= \overline{\psi} y_1 \cdot Hmq.$$

We have

$$\varepsilon^* \overline{m} y_0' = \varepsilon^* \langle T, mgy_0', mg\varepsilon' y_0' \cdot \psi g y_0'\rangle = (\langle X, X', mg\varepsilon' y_0' \cdot \psi g y_0'\rangle, \langle X, T\rangle)$$
$$= (\langle X, X', L\overline{m}\varepsilon' y_0' \cdot L\overline{\psi} g y_0'\rangle, \langle X, T\rangle) = (\overline{m}\varepsilon' \cdot \overline{\psi} g) y_0'$$

and, consequently,

$$\varepsilon^*\overline{m} = \overline{m}\varepsilon' \cdot \overline{\psi}g.$$

Moreover,

$$L\overline{\psi}\cdot\eta^*m = \psi\cdot\eta^*m = m\eta',$$

i.e., by ($1^{\leftarrow}$) $r$ is an object of the category $C$. Let us observe that for any functors

$$m_1, m_1' \colon Y' \to \mathrm{Alm}(A)(X, X')$$

and for any 2-cell

$$\gamma \colon Lm_1 \to Lm_1'$$

there is exactly one 2-cell

$$\overline{\gamma} \colon m_1 \to m_1' \quad \text{with } L\overline{\gamma} = \gamma.$$

Let

$$\tau \colon s = \langle m, \psi\rangle \to s' = \langle m', \psi'\rangle$$

be any morphism in $C'$. By putting $b_X^{\leftarrow}(\tau) = \langle \tau, \overline{\tau g}\rangle$, we obtain a morphism in $C$ from $b_X^{\leftarrow}(s)$ to $b_X^{\leftarrow}(s')$. Indeed, by the second equality of ($2^{\leftarrow}$) we obtain an equality which holds by assumption on $\tau$; moreover, the first equality of ($2^{\leftarrow}$) is obvious. From the definition it follows easily that $b_X^{\leftarrow}$ preserves the composition and that $b_X^{\leftarrow} \colon C' \to C$ is a functor for which $a_X^{\leftarrow}b_X^{\leftarrow} = 1_{C'}$.

To define the unit and counit of an adjunction between the functors $a_X^{\leftarrow}$ and $b_X^{\leftarrow}$ we put $\varepsilon^{0\leftarrow} = \mathrm{id}$ and

$$\begin{aligned}\eta^{0\leftarrow}\langle m, k, \beta, a\rangle = \langle m, \overline{a}\rangle \colon \langle m, k, \beta, a\rangle &\to b_X^{\leftarrow}a_X^{\leftarrow}\langle m, k, \beta, a\rangle \\ &= b_X^{\leftarrow}\langle m, af\cdot L\beta\rangle = \langle m, \overline{m}, \overline{af\cdot L\beta}, \mathrm{id}\rangle.\end{aligned}$$

We can see from ($2^{\leftarrow}$) that $\langle m, \overline{a}\rangle$ is a morphism in $C$, natural in $\langle m, k, \beta, a\rangle$, since for any morphism $\langle \tau, \sigma\rangle$ in $C$ we have

$$\begin{aligned}b_X^{\leftarrow}a_X^{\leftarrow}\langle \tau, \sigma\rangle\cdot\langle m, \overline{a}\rangle &= \langle \tau, \overline{\tau g}\rangle\cdot\langle m, \overline{a}\rangle = \langle \tau, \overline{\tau g}\cdot\overline{a}\rangle = \langle \tau, \overline{\tau g\cdot a}\rangle \\ &= \langle \tau, \overline{a'\cdot L\sigma}\rangle = \langle \tau, \overline{a'}\cdot\sigma\rangle = \langle m', \overline{a'}\rangle\cdot\langle \tau, \sigma\rangle.\end{aligned}$$

Moreover,

$$\begin{aligned}\varepsilon^{0\leftarrow}a_X^{\leftarrow}\langle m, k, \beta, a\rangle\cdot a_X^{\leftarrow}\eta^{0\leftarrow}\langle m, k, \beta, a\rangle &= \varepsilon^{0\leftarrow}\langle m, af\cdot L\beta\rangle\cdot a_X^{\leftarrow}\langle m, \overline{a}\rangle \\ &= m\cdot m = m = a_X^{\leftarrow}\langle m, k, \beta, a\rangle\end{aligned}$$

and

$$\begin{aligned}b_X^{\leftarrow}\varepsilon^{0\leftarrow}\langle m, \psi\rangle\cdot\eta^{0\leftarrow}b_X^{\leftarrow}\langle m, \psi\rangle &= \langle m, \overline{mg}\rangle\cdot\eta^{0\leftarrow}\langle m, \overline{m}, \overline{\psi}, \mathrm{id}\rangle \\ &= \langle m, \overline{m}\rangle\cdot\langle m, \mathrm{id}\rangle = \langle m, \overline{m}\rangle = b_X^{\leftarrow}\langle m, \psi\rangle,\end{aligned}$$

i.e.

$$\varepsilon^{0\leftarrow}a_X\cdot a_X^{\leftarrow}\eta^{0\leftarrow} = a_X^{\leftarrow} \quad \text{and} \quad b_X^{\leftarrow}\varepsilon^{0\leftarrow}\cdot\eta^{0\leftarrow}b_X^{\leftarrow} = b_X^{\leftarrow}.$$

Thus $B_X^{\leftarrow} = \langle a_X^{\leftarrow}, b_X^{\leftarrow}, \eta^{0\leftarrow}, \varepsilon^{0\leftarrow}\rangle$ is an adjunction. Since the counit of this adjunction is identity, $b_X^{\leftarrow}$ is full and faithful.

THEOREM 2.3. *For any object* $g\colon Y' \to Y$ *in* $\mathrm{Kan}(A{\downarrow}A)$, *for any monad* $\boldsymbol{T} = \langle X', T, \eta, \mu\rangle$ *and for any* 0-*cell* $X$ *in* $A$, *let* $\langle T', \varepsilon_1\rangle$ *be the right Kan extension of* $g$ *along* $g$ *with* $\boldsymbol{T}'$ *being the monad of this extension, and let*

$$\overline{m}(y_0') = \langle \boldsymbol{T}, mgy_0', m\varepsilon_1 y_0' \cdot \psi g y_0'\rangle$$

*and*

$$\overline{m}(Q) = (\langle X, X', mgQ\rangle, \langle X, T\rangle)$$

*for any object* $y_0'$ *and any morphism* $Q$ *in* $Y'$. *Then the formulas*

$$s = \langle m, \psi\rangle \xrightarrow{\tau} s' = \langle m', \psi'\rangle \mapsto \langle \overline{m}, m\rangle \xrightarrow{\langle \overline{\tau g}, \tau\rangle} \langle \overline{m}', m'\rangle$$

*define a functor*

$$G\colon 2\text{-}\mathrm{Mon}^{\leftarrow}(A)(\boldsymbol{T}', D_{(X,\boldsymbol{T})}) \to 2\text{-}\mathrm{Kan}(A{\downarrow}A)(g, L_{(X,\boldsymbol{T})})$$

*which is an isomorphism.*

Proof. Since $\langle T', \varepsilon_1\rangle$ is the right Kan extension of $g$ along $g$, $\langle \boldsymbol{T}', gy_0', \varepsilon_1 y_0'\rangle$ is a monad algebra, and thus, by (7), $\overline{m}(y_0')$ is also a monad algebra. Hence $\overline{m}\colon Y' \to \mathrm{Alm}(A)(X, \boldsymbol{T})$ is a functor and $L\overline{m} = mg$. Thus $\langle \overline{m}, m\rangle$ is an object of $2\text{-}\mathrm{Kan}(A{\downarrow}A)(g, L)$, where $L$ is $L_{(X,\boldsymbol{T})}$. Since $L\overline{\tau g} = \tau g$ (see the proof of Theorem 2.2), $\langle \overline{\tau g}, \tau\rangle$ is a morphism of this category from $\langle \overline{m}, m\rangle$ to $\langle \overline{m}', m'\rangle$. Obviously, $G$ preserves the composition, and thus $G$ is a functor. Consider the functor

$$\mathrm{Str}\colon \mathrm{Kan}(A{\downarrow}A) \to \mathrm{Mon}^{\leftarrow}(A)$$

defined in Theorem 1.4.4 in [5]. Now a functor

$$G_1\colon 2\text{-}\mathrm{Kan}(A{\downarrow}A)(g, L_{(X,\boldsymbol{T})}) \to 2\text{-}\mathrm{Mon}^{\leftarrow}(A)(\boldsymbol{T}', D_{(X,\boldsymbol{T})})$$

is well defined by the formula

$$G_1(\langle m, m'\rangle) = \mathrm{Str}(\langle m, m'\rangle) = \langle m', \gamma\rangle$$

on objects and by

$$G_1(\langle \sigma, \tau\rangle) = \tau$$

on morphisms. Indeed, let

$$\langle \sigma, \tau\rangle\colon \langle m, m'\rangle \to \langle m_1, m_1'\rangle$$

be any morphism in the first category. Then $\tau = G_1(\langle \sigma, \tau\rangle)$ is a morphism in the second category from $\mathrm{Str}(\langle m, m'\rangle) = \langle m', \gamma\rangle$ to $\mathrm{Str}(\langle m_1, m_1'\rangle) = \langle m_1', \gamma_1\rangle$. Indeed (see the proof of Theorem 1.4.4 in [5]),

$$\begin{aligned} a_1\cdot(z_1\cdot\tau T'\cdot\gamma)g &= a_1\cdot z_1 g\cdot\tau T' g\cdot\gamma g = m_1'\varepsilon_1\cdot\tau T' g\cdot\gamma g = (\tau\varepsilon_1)\cdot\gamma g \\ &= \tau g\cdot m'\varepsilon_1\cdot\gamma g = L\sigma\cdot a\cdot zg\cdot z^{-1}g\cdot hg = L\sigma\cdot a\cdot hg = L\sigma\cdot\varepsilon^* m = \varepsilon^*\sigma \\ &= \varepsilon^* m_1\cdot DL\sigma = a_1\cdot h_1 g\cdot D\tau g = a_1\cdot(z_1\cdot\gamma_1\cdot D\tau)g, \end{aligned}$$

whence

$$z_1 \cdot \tau T' \cdot \gamma = z_1 \cdot \gamma_1 \cdot D\tau,$$

and thus

$$\tau T' \cdot \gamma = \gamma_1 \cdot D\tau,$$

where $D$ is $D_{(X,T)}$. Obviously, $G_1$ preserves the composition and $GG_1 = 1$, $G_1 G = 1$. Thus $G_1$ is the inverse functor of $G$, and $G$ is an isomorphism.

For $X = 1$ and $g = L_{(1,T_1)}$, Theorem 2.3 gives a result of Alagic [1].

**3. Characterization of monadic functors.** Let $A = 2\text{-CAT}(U)$, where $U$ is any universe. In this case the functor $\mathrm{Sem}(X, A)$ defined in 1.6 is of the form

$$\mathrm{Sem}(X, A)\colon \mathrm{Mon}^{\leftarrow}(A) \to \mathrm{Kan}(A{\downarrow}A).$$

Let $X = 1$ be the one-object category in $A$. Consider the functor

$$\mathrm{Str}\colon \mathrm{Kan}(A{\downarrow}A) \to \mathrm{Mon}^{\leftarrow}(A)$$

defined in Theorem 1.4.4 in [5].

THEOREM 3.1. *The functor* $\mathrm{Str}\colon \mathrm{Kan}(A{\downarrow}A) \to \mathrm{Mon}^{\leftarrow}(A)$ *is left adjoint to the functor*

$$\mathrm{Sem}(1, A)\colon \mathrm{Mon}^{\leftarrow}(A) \to \mathrm{Kan}(A{\downarrow}A).$$

Proof. We define a natural transformation

$$\varphi\colon \mathrm{Kan}(A{\downarrow}A) \to \mathrm{Sem}(1, A)\mathrm{Str}$$

by putting, for any object $g\colon X' \to X$ in $\mathrm{Kan}(A{\downarrow}A)$,

$$\varphi g = \langle K_g, X\rangle,$$

where $K_g\colon X' \to \mathrm{Alm}(A)(1, T)$ is the comparison functor of $g$, i.e. $\langle T, \varepsilon\rangle$ is the right Kan extension of $g$ along $g$ and $K_g(x') = \langle T, gx', \varepsilon x'\rangle$ on objects, $K_g(Q) = gQ$ on morphisms. For each morphism $\langle m, m'\rangle\colon g \to g'$ in $\mathrm{Kan}(A{\downarrow}A)$ we have

$$\begin{aligned}\mathrm{Sem}(1, A)\mathrm{Str}(\langle m, m'\rangle)\cdot\varphi g &= \langle \mathrm{al}(\langle m', \gamma\rangle), m'\rangle \cdot \langle K_g, X\rangle \\ &= \langle \mathrm{al}(\langle m', \gamma\rangle)K_g, m'\rangle = \langle K_{g'} m, m'\rangle = \varphi g' \cdot \langle m, m'\rangle,\end{aligned}$$

since

$$\begin{aligned}\mathrm{al}(\langle m', \gamma\rangle)K_g x' &= \mathrm{al}(\langle m', \gamma\rangle)(\langle T, gx', \varepsilon x'\rangle) = \langle T', m'gx', m'\varepsilon x' \cdot \gamma gx'\rangle \\ &= \langle T', g'mx', m\varepsilon' x' \cdot z^{-1} gx' \cdot hgx'\rangle = \langle T', g'mx', ax' \cdot hgx'\rangle \\ &= \langle T', g'mx', \varepsilon' mx'\rangle = K_{g'} mx'\end{aligned}$$

for each object $x'$ in $X$. For any object $T_1$ in $\mathrm{Mon}^{\leftarrow}(A)$ and for any morphism $\langle m, m'\rangle\colon g \to L_{(1,T_1)}$ in $\mathrm{Kan}(A{\downarrow}A)$ there is only one morphism $\langle m', \xi\rangle\colon T \to T_1$ in $\mathrm{Mon}^{\leftarrow}(A)$ such that

$$\langle m', \xi\rangle \cdot \varphi g = \langle m, m'\rangle.$$

We define $\xi$ as $z^{-1}\cdot w$, where

$$\langle X', w\rangle : \langle Tm', pm\rangle \to \langle T'', a\rangle$$

is the unique morphism in the category $2\text{-}dA(X, -)(g, m'g)$ from $\langle Tm', pm\rangle$ to the terminal object $\langle T'', a\rangle$ of this category, and

$$\langle X', z\rangle : \langle T'', a\rangle \xrightarrow{\sim} \langle m'T, m'\varepsilon\rangle$$

is an isomorphism in this category to the right Kan extension of $m'g$ along $g$ (see [5]). Moreover,

$$p: TL_{(1,T)} \to L_{(1,T)}$$

is the natural transformation defined by $p\langle T, x, q\rangle = q$. Hence Str is left adjoint to $\mathrm{Sem}(1, A)$ with unit $\varphi$. Let us observe that

$$\mathrm{Str}\,\mathrm{Sem}(1, A) = 1_{\mathrm{Mon}^{\leftarrow}(A)},$$

since

$$\mathrm{Str}\,\mathrm{Sem}(1, A)T = \mathrm{Str}\,L_{(1,T)} = T$$

and

$$\mathrm{Str}\,\mathrm{Sem}(1, A)(\langle m, \sigma\rangle) = \mathrm{Str}\big(\langle \mathrm{al}(\langle m, \sigma\rangle), m\rangle\big) = \langle m, \gamma\rangle \qquad \text{with } \sigma = \gamma$$

by the definition of Str (see [5]) and by the equality

$$p'\,\mathrm{al}(\langle m, \sigma\rangle)(\langle T, x, q\rangle) = mp\cdot\sigma L_{(1,T)}(\langle T, x, q\rangle).$$

Hence the counit of this adjunction is identity, and thus the functor $\mathrm{Sem}(1, A)$ is full and faithful. Therefore, we have the adjunction

$$\langle \mathrm{Str}, \mathrm{Sem}(1, A), \varphi, \mathrm{id}\rangle : \mathrm{Kan}(A{\downarrow}A) \to \mathrm{Mon}^{\leftarrow}(A)$$

which defines a monad $S = \langle \mathrm{Kan}(A{\downarrow}A), S, \varphi, \mu^{(S)}\rangle$ such that $\mu^{(S)}$ is a natural isomorphism. It follows that $\varphi L_{(1,S)}$ is a natural isomorphism with the inverse

$$p: SL_{(1,S)} \to L_{(1,S)}.$$

Thus we have the following algebraic characterization of monadic functors:

THEOREM 3.2. *Let $g: X' \to X$ be any object of* $\mathrm{Kan}(A{\downarrow}A)$ *and let $T$ be the monad defined by $g$. Then the comparison functor*

$$K_g: X' \to \mathrm{Alm}(A)(1, T)$$

*is an isomorphism if and only if there is exactly one $S$-structure on $g$, i.e. if and only if there is exactly one morphism $\langle m, m'\rangle : Sg \to g$ in* $\mathrm{Kan}(A{\downarrow}A)$ *such that $\langle S, g, \langle m, m'\rangle\rangle$ is a monad $S$-algebra in the category* $\mathrm{Kan}(A{\downarrow}A)$.

Proof. Let $\langle S, g, \langle m, m'\rangle\rangle$ be any monad $S$-algebra in $\mathrm{Kan}(A{\downarrow}A)$. Then

$$\varphi L_{(1,S)}\langle S, g, \langle m, m'\rangle\rangle = \varphi g = \langle K_g, X\rangle$$

is an isomorphism in $\mathrm{Kan}(A{\downarrow}A)$ with the inverse $\langle m, m'\rangle$. From the definition of the monad $S$-algebra it follows that $m' = X$ and $K_g m = X'$, that is, $K_g$ is an isomorphism in $A$ with the inverse functor $m$. Let $K_g: X' \to \mathrm{Alm}(A)(1, T)$ be an isomorphism in $A$ with the inverse functor $m: \mathrm{Alm}(A)(1, T) \to X'$. Then $\langle S, g, \langle m, X\rangle\rangle$ is a unique monad $S$-algebra in $\mathrm{Kan}(A{\downarrow}A)$ on $g$.

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