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CLOSURE AND INTERIOR IN FINITE TOPOLOGICAL SPACES

BY

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§ 1. Introduction. Kuratowski [2] has shown that at most 14 distinct sets can be constructed from a subset of a topological space by application of the closure, complement, and interior operators in any order. We will say that a set A is a *K-set* if exactly 14 distinct sets can be constructed from A in this manner. A topological space is said to be a *K-space* if it contains a K-set. If B is the set in the real line consisting of all points in $(0,1) \cup (1,2)$, all rationals in $(2, 3)$ and the point 4, then it is well known that B is a K-set.

In this paper an example will be given of a finite topological space having seven points which is a K-space. We will show that no topological space having less than seven points can be a K-space. However, the topology of a seven point K-space is not unique. In fact, there are five non-homeomorphic topologies on a seven-point set which make it a K-space.

How much structure can we impose on a finite topological space and still have it be a K-space? If we require T_1 (singleton sets are closed; [1]), then every set is a finite union of closed sets and therefore closed. Hence, the topology is discrete and there are no K-sets. We will show that if we require T_0 (given any two distinct points there exists a neighborhood of one not containing the other; [1]), then at most 10 distinct sets can be constructed. That it is possible to have exactly 10 distinct sets is shown by an example in a five-point T_0-space. We go on to show that no T_0-space having less than five elements can contain a subset from which 10 distinct sets can be constructed. The topology on the five-point space is not unique, however, and we exhibit three non-homeomorphic T_0 topologies which satisfy the requirement.

We are greatly indebted to Dr. H. Nakano for his advice and encouragement during the preparation of this paper.

In the sequel we will use the notation A^-, A° and A' for the closure, interior, and complement of a subset A of a topological space according to [3].

§ 2. Finite topological spaces. We begin with

EXAMPLE 1 (due to H. Nakano). Let $S = \{1, 2, 3, 4, 5, 6, 7\}$ and let τ_0 be a topology on S with base β_0 given by

$$\beta_0 = \{\emptyset, S, \{1\}, \{7\}, \{1, 2\}, \{6, 7\}, \{3, 5\}\}.$$

Then $A = \{1, 3, 6\}$ is a K-set

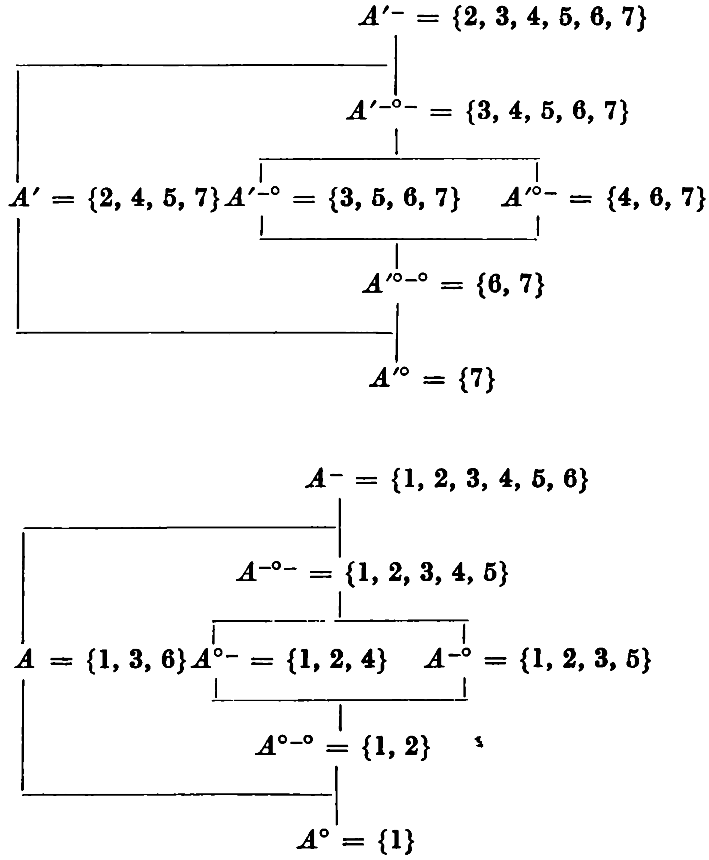

Fig. 1

In figure 1 if two sets are connected by a line then the set below is contained in the one above. Verification of the accuracy of the figure is a straightforward exercise.

LEMMA 1. *If* $A^{\circ-} \supset A^{-\circ}$, *then* $A^{\circ-} = A^{-\circ-}$ *and* $A^{-\circ} = A^{\circ-\circ}$.

Proof. $A^{\circ-} \supset A^{-\circ-} \supset A^{\circ-}$ and $A^{\circ-\circ} \supset A^{-\circ} \supset A^{\circ-\circ}$.

LEMMA 2. *For any subset* A *of an arbitrary topological space* S (*not necessarily finite*) $A^{-\circ} \cap A^{\circ-\prime}$ *cannot contain a singleton open set.*

Proof. Assume $\{x\} = \{x\}^{\circ}$. Necessarily either $x \in A$ or $x \in A'$. $x \in A$ implies $x \in A^{\circ} \subset A^{\circ-}$ which implies $x \notin A^{-\circ} \cap A^{\circ-\prime}$. $x \in A'$ implies $x \in A'^{\circ} \subset \subset A'^{\circ-} = A^{-\circ\prime}$ which implies $x \notin A^{-\circ} \cap A^{\circ-\prime}$.

THEOREM 1. *No topological space having less than seven points can be a K-space.*

Proof. Assume A is a K-set in a finite topological space. Since $A^{\circ -}$ and $A^{-\circ -}$ are distinct by the definition of a K-set, then, by Lemma 1, $A^{\circ -} \not\supset A^{-\circ}$ or, in other words, $A^{\circ -\prime} \cap A^{-\circ} = G \neq \emptyset$. Also, since G is open, Lemma 2 shows that G is not a singleton. Therefore, G contains at least two points or, equivalently, $A^{-\circ}$ contains at least two points (call them x and y), which are not contained in $A^{\circ -}$. Moreover, $A^{\circ -}$ properly contains $A^{\circ - \circ}$ and therefore must contain at least one more point (call it w) than $A^{\circ - \circ}$. Thus, if we try to build up from A° adding the minimum number of points consistent with the above requirements and with the requirement that the sets be distinct, we are led to figure 2.

$$A^{-} = \{a_1, \dots, a_n, a_{n+1}, w, x, y, z\}$$

$$A^{-\circ -} = \{a_1, \dots, a_n, a_{n+1}, w, x, y\}$$

$$A^{\circ -} = \{a_1, \dots, a_n, a_{n+1}, w\} \qquad A^{-\circ} = \{a_1, \dots, a_n, a_{n+1}, x, y\}$$

$$A^{\circ - \circ} = \{a_1, \dots, a_n, a_{n+1}\}$$

$$A^{\circ} = \{a_1, \dots, a_n\}$$

Fig. 2

We see that if A° has n elements A^- has a minimum of $n+5$ elements. Since either $A^\circ = \emptyset$ or $A^- = S$ would mean that A is not a K-set, n must be at least 1 and $n+5$ must be less than the number of elements in S. Choosing $n = 1$ we find that the space must have at least seven points.

The above reasoning also shows that in a seven-point K-space there are exactly two singleton open sets, the interior of any K-set and the interior of its complement.

THEOREM 2. *There are exactly five non-homeomorphic topologies on a seven-point set which make the set a K-space.*

Proof. Let $R = \{a_1, \dots, a_7\}$. If (R, τ) is a K-space and A is a K-set in R, then A° and A'° are the only two singleton open sets. Also $A^{\circ - \circ}$ and $A'^{\circ - \circ}$ must be two-point open sets. Furthermore, $A^{\circ - \prime} \cap A^{-\circ} = G$ is a two-point open set disjoint from $A^{\circ - \circ}$ and from $A'^{\circ - \circ} = A^{-\circ -\prime}$. Define

$$f\colon S = \{1, 2, 3, 4, 5, 6, 7\} \to R$$

by: $f(1) =$ the only element of A°; $f(7) =$ the only element of A'°; $f(2) =$ the only element of $A^{\circ - \circ} - A^\circ$; $f(6) =$ the only element of $A'^{\circ - \circ} - A'^\circ$;

$f(3)$ = one of the two elements in G; $f(5)$ = the other element in G; $f(4)$ = the remaining element in R. Now define a topology τ' on S by: X is open in S if and only if $f(X)$ is open in R. We see that (S, τ') is homeomorphic to (R, τ) and τ' is stronger than τ_0, the topology of example 1. Thus, any seven-point K-space is homeomorphic to (S, τ'), where τ' is stronger than τ_0.

Now suppose $B \in \tau'$ and $B \notin \tau_0$. If

$$B \cap \{3, 5\} = B \cap (A^{-\circ} \cap A^{\circ-\prime}) \neq \emptyset$$

we must have $B \supset \{3, 5\}$ since otherwise $B \cap \{3, 5\}$ would be a singleton open set contradicting Lemma 2. Suppose $7 \notin B$. Then $B \subset A^-$ which means, since B is open, $B \subset A^{-\circ} = \{1, 2, 3, 5\}$. Either $B \cap \{3, 5\} = \emptyset$ or $B \supset \{3, 5\}$ leads to $\{2\} \in \tau'$, a contradiction with the fact that there are exactly two singleton open sets in a seven-point K-space. Since both possibilities lead to a contradiction we must have $7 \in B$. A symmetric argument shows $1 \in B$. Thus, $B \in \tau'$ and $B \notin \tau_0$ implies $B \supset \{1, 7\}$ and either $B \supset \{3, 5\}$ or $B \cap \{3, 5\} = \emptyset$.

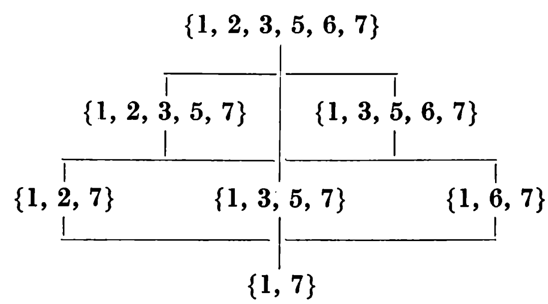

Fig. 3

Inspection shows that figure 3 contains all possible candidates for new open sets which do not contain the number 4. However, all these sets are open in the original topology τ_0; hence they give nothing new. Thus, we will get all possible candidates by inserting the number 4 at various places in figure 3. Moreover, any topology obtained in this manner will give us a K-space. In fact, all new open sets will contain both the number 1 and the number 7 and thus can never be contained in any of the fourteen sets constructed from the K-set A. Hence, the operation of taking the interior of a set will lead to the same result in all cases and since $A^- = A'^{\circ\prime}$ we see that the closure operation will give the same sets as before.

Now, referring again to figure 3, if we insert a 4 in the set in the top row we merely get the original topology. Inserting a 4 on the left side in the second row from the top gives a second topology. A third topology results if a four is placed in the set on the left side in the third row down. If a 4 is put in the set in the middle of the third row from the top we get a fourth topology. Finally, a 4 in the set on the bottom gives us a fifth topology. As we go up, following the lines in the figure each set above is the union of the set below with an open set. Thus, if $\{1, 4, 7\}$ is open, then a 4 inserted in *every* set of the diagram also gives an open set.

Placing more than one 4 at a time gives nothing new since the intersection of two open sets is open. Furthermore, we get nothing new by placing 4's on the right side since any topology obtained in that manner is homeomorphic to the topology obtained by inserting a 4 in the corresponding position on the left side. The required homeomorphism is the mapping which trades 1 for 7 and 2 for 6 and leaves the others unchanged. This is clearly a homeomorphism since the basis sets of τ_0 are merely permuted and an open set in the diagram on the right side is mapped onto its symmetric counterpart on the left side.

§ 3. Finite T_0-topological spaces. The following result is easily proved by induction.

LEMMA 3. *Every non-empty finite T_0-space contains a singleton open set.*

THEOREM 3. *In a finite T_0-space, $A^{-\circ} = A^{\circ-\circ}$ and $A^{\circ-} = A^{-\circ-}$.*

Proof. $A^{-\circ} \cap A^{\circ-\prime}$ is an open subset of the T_0-space and, by Lemma 3, if it is non-empty it must contain a singleton open set since it is a T_0-space itself. This is impossible by Lemma 2 so we have $A^{-\circ} \cap A^{\circ-\prime} = \emptyset$ or $A^{\circ-} \supset A^{-\circ}$. Then Lemma 1 shows that $A^{-\circ} = A^{\circ-\circ}$ and $A^{\circ-} = A^{-\circ-}$. (Also, of course, $A'^{-\circ} = A'^{\circ-\circ}$ and $A'^{\circ-} = A'^{-\circ-}$.)

Thus, we see that a finite T_0-space cannot be a K-space and, in fact, no more than ten distinct subsets can be constructed from any subset.

EXAMPLE 2. Let $S = \{1, 2, 3, 4, 5\}$ and let τ_1 be a topology on S with base β_1 given by: $\beta_1 = \{\emptyset, S, \{1\}, \{5\}, \{1, 2\}, \{4, 5\}\}$.

Then (S, τ_1) is a T_0-space and, referring to figure 4, we see that ten distinct sets can be constructed from $A = \{1, 3, 4\}$

$$A'^{-} = \{2, 3, 4, 5\} \qquad\qquad A^{-} = \{1, 2, 3, 4\}$$

$$A'^{\circ-} = \{3, 4, 5\} \qquad\qquad A^{\circ-} = \{1, 2, 3\}$$

$$A' = \{2, 5\} \qquad\qquad A = \{1, 3, 4\}$$

$$A'^{-\circ} = \{4, 5\} \qquad\qquad A^{-\circ} = \{1, 2\}$$

$$A'^{\circ} = \{5\} \qquad\qquad A^{\circ} = \{1\}$$

Fig. 4

Figure 4 is easily verified. The inclusion relations among the sets hold true in any finite T_0-space. Hence, is it obvious from the diagram that no set with fewer than five points would fill the requirements.

THEOREM 4. *There are three non-homeomorphic T_0-topologies on a five-point space such that ten distinct sets can be constructed from a subset of the resulting topological space by application of the closure, complement and interior operators in any order.*

Proof. Just as in the proof of Theorem 2 it is easily shown that τ_1 is the weakest such topology (up to homeomorphism). In considering possibilities for stronger topologies we arrive at figure 5.

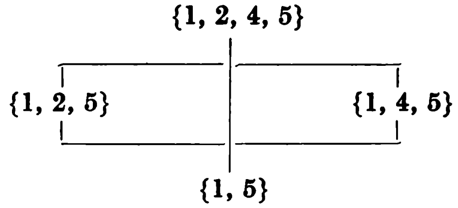

Fig. 5

Inserting the number 3 in the top, left hand and bottom sets we get three non-homeomorphic topologies.

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