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EXTREME OPERATORS ON 2-DIMENSIONAL l_p-SPACES

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For any two Banach spaces E and F we denote by $\mathscr{L}(E, F)$ the Banach space of all linear bounded operators from E into F. An operator $T \in \mathscr{L}(E, F)$ is called a *contraction* if $\|T\| \leqslant 1$; T is called an *extreme contraction* if T is an extreme point of the unit ball of $\mathscr{L}(E, F)$.

Let $1 \leqslant p \leqslant \infty$ and let A be a non-empty index set. As usual, we denote by $l_p(A)$ the Banach space of all p-summable functions on A. We denote by e_i the element of $l_p(A)$ defined by $e_i(j) = \delta_{ij}$ for $i, j \in A$.

If $p = \infty$, then using the well-known Banach lattice isomorphism $l_\infty(A) \to C(\beta A)$ and Sharir's result in [7] we can characterize all extreme contractions in $\mathscr{L}\big(l_\infty(A), l_\infty(A)\big)$ as the lattice homomorphisms taking 1 into 1, multiplied by functions of absolute value 1 (see also [1] and [3]).

For $p = 1$ and real l_1-spaces, the extreme contractions can be characterized (by duality) as those operators whose adjoints are extreme contractions in $\mathscr{L}\big(l_\infty(A), l_\infty(A)\big)$ (see [1], Proposition, and [3] for $|A| = \aleph_0$).

In case of $p = 2$ and complex scalars, the set of extreme contractions coincides with the set of all isometries and coisometries of the Hilbert space l_2 (see [2], Theorem 1). Throughout the paper we assume that $1 < p < \infty$.

LEMMA 1. *Let $p > 2$ and let $T = e_i \otimes y \in \mathscr{L}\big(l_p(A), l_p(A)\big)$, $i \in A$. If $\|y\| = 1$ and $y(j) \neq 0$ for all $j \in A$, then T is an extreme contraction.*

Proof. Let $T = e_i \otimes y$, i.e. $Tx = \langle x, e_i \rangle y$ and $\|T \pm R\| \leqslant 1$ for some $R \in \mathscr{L}\big(l_p(A), l_p(A)\big)$. If $p > 2$, then $1 < q < 2$ whenever $1/p + 1/q = 1$. By the uniform convexity of $l_p(A)$ we have $Re_i = 0$, so

$$\langle e_i, R'x \rangle = \langle Re_i, x \rangle = 0 \quad \text{for all } x \in l_q(A).$$

Now we put

$$y^1 = \sum_{j \in A} \operatorname{sign} y(j)\, |y(j)|^{p-1} e_j \in l_q(A).$$

By an easy calculation we observe that $\|y^1\| = 1$ and $\|T' y^1\| = e_i$ Therefore, $\|T' y^1\| = \|y^1\|$, so $R' y^1 = 0$ by the uniform convexity of $l_q(A)$.

For $\varepsilon > 0$ and $k \in A$ we have

$$\|(T' \pm R')(y^1 + \varepsilon \operatorname{sign} y(k) e_k)\| \leqslant \|y^1 + \varepsilon \operatorname{sign} y(k) e_k\|.$$

Therefore, since $T' e_k = y(k) e_i$, we get

$$\|e_i(1 + \varepsilon |y(k)|)\|^q + \varepsilon^q \|R' e_k\|^q \leqslant \sum_{j \neq k} |y(j)|^p + |\operatorname{sign} y(k)(|y(k)|^{p-1} + \varepsilon)|^q.$$

Hence, we obtain

$$\|R' e_k\|^q \leqslant \varepsilon^{-q}[1 - |y(k)|^p + (|y(k)|^{p-1} + \varepsilon)^q - (1 + \varepsilon |y(k)|)^p].$$

Using twice the de l'Hospital rule we can see that the right-hand side of this inequality tends to 0 as $\varepsilon \to 0$. Therefore, $R' e_k = 0$, so $R = 0$, which completes the proof of the lemma.

Let X be the 2-dimensional real l_p-space, $1 < p < \infty$. Clearly, X can be identified with $\boldsymbol{R}^2$ endowed with the p-norm $\|(x_1, x_2)\| = (|x_1|^p + |x_2|^p)^{1/p}$. We denote by $\mathscr{S}$ the unit sphere in $\mathscr{L}(X, X)$, and by U the unit ball in X.

For any $x = (x_1, x_2) \in \boldsymbol{R}^2$ we write

$$x^1 = (\operatorname{sign} x_1 |x_1|^{p-1}, \operatorname{sign} x_2 |x_2|^{p-1}) \quad \text{and} \quad x^0 = (-x_2, x_1).$$

Note that if $\|x\| = 1$ in X, then x^1, considered as an element of thg dual Banach space X' of X, is the unique functional of norm 1 attaininl its norm on x. Also $(x^1)^0 = (x^0)^1$ holds and $(x^0)^1 \in X'$ is a unique functiona of norm 1 attaining its norm on $x^0 \in X$. On the other hand, x^0 as an element of X' is a (unique up to a scalar multiple) functional annihilating x. Let us note that if $x \neq 0$, then x and $(x^0)^1$ are linearly independent.

Let $x, y \in X$ be fixed with $\|x\| = \|y\| = 1$ and let

$$\mathscr{A}_{x,y} = \{T \in \mathscr{L}(X, X) : Tx = y\}.$$

Observe that $x^1 \otimes y \in \mathscr{A}_{x,y}$, so $\mathscr{A}_{x,y} \neq \emptyset$. Clearly, all operators in $\mathscr{A}_{x,y}$ have norms greater than or equal to 1. It is easy to see that $\mathscr{A}_{x,y}$ contains all operators of the form

$$T_t = x^1 \otimes y + t x^0 \otimes (y^0)^1, \qquad t \in \boldsymbol{R}.$$

We claim the set $\mathscr{I}_{x,y} = \mathscr{A}_{x,y} \cap \mathscr{S}$ consists of exactly such operators. Indeed, for $S, T \in \mathscr{I}_{x,y}$ we have $(S - T)x = 0$, whence $\dim(\operatorname{range}(S - T)) \leqslant 1$ $(\dim X = 2)$, and so $S - T = x^0 \otimes z$ for some $z \in X$. Since $S' y^1 = T' y^1 = x^1$, we have also $(S - T)' y^1 = 0$ and, analogously, $(S - T)' = (y^0)^1 \otimes w$ for some $w \in X'$. Thus $S - T = w \otimes (y^1)^0$, which implies $z = t(y^1)^0$ for some $t \in \boldsymbol{R}$ and proves our claim. Obviously, the operator T_t is a linear automorphism of X for $t \neq 0$.

We infer that the set $\mathscr{I}_{x,y}$ is a closed non-empty (possibly degenerated) line segment in $\mathscr{L}(X, X)$. Let us observe that $\mathscr{I}_{x,y}$ is a face (i.e. an extreme convex closed subset) of the unit ball in $\mathscr{L}(X, X)$. In fact, for any $T \in \mathscr{I}_{x,y}$ and $R \in \mathscr{L}(X, X)$ with $\|T \pm R\| \leqslant 1$ we have $\|(T \pm R)x\| = \|y \pm Rx\| \leqslant 1$, whence $Rx = 0$ by the uniform convexity of X. This implies $T \pm R \in \mathscr{A}_{x,y}$ and, consequently, $T \pm R \in \mathscr{I}_{x,y}$. Finally, let us note that any operator $T \in \mathscr{S}$ attains its norm on a vector x of norm 1. Hence $\mathscr{S}$ is a union of the sets $\mathscr{I}_{x,y}$, $\|x\| = \|y\| = 1$. Thus we have the following

PROPOSITION. *For any pair x, y of unit vectors in X the set*

$$\mathscr{I}_{x,y} = \{T \in \mathscr{S}\colon\ Tx = y\}$$

is a closed 0- or 1-dimensional face of the unit ball in $\mathscr{L}(X, X)$. Moreover, $\mathscr{S} = \bigcup \mathscr{I}_{x,y}$, where the union extends over all pairs x, y with $\|x\| = \|y\| = 1$.

In particular, $\mathscr{S}$ is the 1-skeleton of the unit ball in $\mathscr{L}(X, X)$ (we recall that the k-skeleton of a convex set Q is the set of all points $x \in Q$ such that the face generated by x has dimension less than or equal to k; see [6]).

Let $p > 2$ and $\|x\| = \|y\| = 1$. For every $t \in \boldsymbol{R}$ we define a function $f_t\colon \boldsymbol{R} \to \boldsymbol{R}$ by

$$f_t(\tau) = \frac{\|T_t(x+\tau(x^0)^1)\|^p}{\|x+\tau(x^0)^1\|^p}.$$

Obviously, $f_t(0) = 1$ for every $t \in \boldsymbol{R}$. Moreover, f_t has the second derivative and by a standard calculation we obtain

$$f_t'(0) = 0 \quad \text{and} \quad f_t''(0) = p(p-1)[t^2|y_1y_2|^{p-2} - |x_1x_2|^{p-2}]$$

for every $t \in \boldsymbol{R}$. If $x_1x_2y_1y_2 \neq 0$, then the third derivative of f_t exists and we have

$$f_t'''(0) = p(p-1)(p-2)[t^3 \operatorname{sign}(y_1y_2)|y_1y_2|^{p-3}(|y_1|^p - |y_2|^p) - \\ - \operatorname{sign}(x_1x_2)|x_1x_2|^{p-3}(|x_1|^p - |x_2|^p)].$$

If $T_t \in \mathscr{I}_{x,y}$, then, obviously, $f_t(\tau) \leqslant 1$. Since $f_t(0) = 1$, we have $f_t'(0) = 0$ and $f_t''(0) \leqslant 0$. Moreover, if $f_t''(0) = 0$ and the third derivative exists, then also $f_t'''(0) = 0$.

LEMMA 2. *Let $p \geqslant 2$, $x_1x_2y_1y_2 \neq 0$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$, and let T_s be an end point of $\mathscr{I}_{x,y}$. If $f_s''(0) = 0$, then T attains its norm on two linearly independent vectors.*

Proof. Obviously, $f_s'''(0) = 0$. Thus we obtain the following pair of equalities:

$$s^6|y_1y_2|^{3p-6} = |x_1x_2|^{3p-6}, \tag{1}$$

$$s^3\operatorname{sign}(y_1y_2)|y_1y_2|^{p-3}(|y_1|^p - |y_2|^p) = \operatorname{sign}(x_1x_2)|x_1x_2|^{p-3}(|x_1|^p - |x_2|^p). \tag{2}$$

Now, by adding both sides of the square of (2) to both sides of (1) multiplied by 4, we obtain

$$s^6|y_1y_2|^{2p-6} = |x_1x_2|^{2p-6}. \tag{3}$$

By (1) and (3) we have $|x_1 x_2| = |y_1 y_2|$ and $|s| = 1$. Thus by a standard calculation we have $|x_1| = |y_1|$ or $|x_1| = |y_2|$, so $\|(x^0)^1\| = \|(y^0)^1\|$. Therefore, T_s attains its norm on both x and $(x^0)^1$.

LEMMA 3. *Let $p \geqslant 2$ and let T_s be an end point of $\mathscr{I}_{x,y}$. If $f_s''(0) < 0$, then T attains its norm on two linearly independent vectors.*

Proof. We consider the case of $s \geqslant 0$. By the continuity of $f_t''(0)$ as a function of t, there exists $r > s$ with $f_r''(0) < 0$. Since $f_r'(0) = 0$ and $f_r''(0) < 0$, there exists $\varepsilon > 0$ such that if $|\tau| < \varepsilon$, then $f_r(\tau) \leqslant 1$. Let

$$V = \{a(x + \beta(x^0)^1) : |\beta| < \varepsilon,\ a \in \boldsymbol{R}\}.$$

Therefore, if $a(x+\beta(x^0)^1) \in V$, then $f_r(\beta) \leqslant 1$, so $\|T_r v\| \leqslant \|v\|$ for $v \in V$. The set

$$T_r(V) = \{a(y + r\beta(y^0)^1) :\ |\beta| < \varepsilon,\ a \in \boldsymbol{R}\}$$

is open, since the vectors y and $(y^0)^1$ are linearly independent and $r > 0$ (in particular, T_r is one-to-one).

Let us note that if $0 \leqslant t' < t$, then $T_{t'}(U) \subset T_t(U)$, where U is the unit ball of X. Indeed, let $u \in T_{t'}(U)$. Then there exists $v \in U$ such that

$$u = y\langle v, x^1\rangle + t'(y^0)^1\langle v, x^0\rangle.$$

Also $u = T_t w$, where $w = (t'/t)v + (1 - t'/t)\langle v, x^1\rangle x$ and $w \in U$ by the triangle inequality.

Let $K_t = T_t(U) \setminus (\operatorname{Int} U \cup T_r(V))$. The sets K_t are compact and $K_{t'} \subset K_t$ if $0 \leqslant t' < t$.

If $s < t < r$, then $T_t z_t \in K_t$, where z_t is a unit vector on which T_t attains its norm. Indeed, we have $T_t z_t \in T_t(U) \subset T_r(U)$. If $T_t z_t \in T_r(V)$, then $T_t z_t \in T_r(U \cap V)$ (T_r is one-to-one) and there exists $v \in S \cap V$ such that $T_t z_t = T_r v$. Since T_s is an end point of $\mathscr{I}_{x,y}$, we have

$$1 < \|T_t\| = \|T_t z_t\| = \|T_r v\| \leqslant \|v\| \leqslant 1.$$

From this contradiction it follows that $T_t z_t \notin T_r(V)$. Thus $T_t z_t \in K_t$. Therefore, the K_t's are non-empty for $s < t < r$, and by compactness there exists a vector z with $z \in K_t$ for all $s < t < r$. Since $1 \leqslant \|z\| \leqslant \|T_t\|$ for all $s < t < r$, we have $\|z\| = 1$. Also, for every t ($s < t < r$), $z = T_t x_t$ for some $x_t \in U$. By the continuity of $t \to T_t$ and by compactness, $z = T_s x_s$ for certain $x_s \in U$. Since $T x_s = z \neq \pm y \in T_r(V)$, the vectors x_s and x are linearly independent.

In case of $s \leqslant 0$ the proof is analogous.

THEOREM. *Let $1 < p < \infty$, $p \neq 2$, and $T \in \mathscr{L}(X, X)$. Then T is an extreme contraction if and only if either T attains its norm on two linearly* *independent vectors in X or T is of the form*

$$T = \begin{cases} x\otimes e_i & \text{for } 1 < p < 2, \\ e_i\otimes y & \text{for } 2 < p < \infty \end{cases}$$

with $x, y \neq \pm e_j$ $(i, j = 1, 2)$, $\|x\| = \|y\| = 1$.

Proof. Assume $p > 2$. If a contraction on X attains its norm on two linearly independent vectors, it is obviously extreme (since $\dim X = 2$). If a contraction T is of the form $T = e_i\otimes y$, where $y \neq \pm e_j$ $(i, j = 1, 2)$ and $\|y\| = 1$, then by Lemma 1 it is also extreme.

Conversely, if T is an extreme contraction, then $\|T\| = 1$, so there exist $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with $\|x\| = \|y\| = 1$, and $Tx = y$. We consider 4 cases:

1. If $x_1x_2 \neq 0$ and $y_1y_2 \neq 0$, then the assertion follows from Lemma 2 if $f_s''(0) = 0$ or from Lemma 3 if $f_s''(0) < 0$.

2. If $x_1x_2 \neq 0$ and $y_1y_2 = 0$, then $f_s''(0) < 0$ and we can again apply Lemma 3.

3. If $x_1x_2 = 0$ and $y_1y_2 \neq 0$, then $s = 0$ since $f_s''(0) \leqslant 0$ for $T_s \in \mathscr{I}_{x,y}$. Thus $\mathscr{I}_{x,y}$ is degenerated. Moreover, $T_s = e_i\otimes y$ with $y \neq \pm e_j$ $(i, j = 1, 2)$, since from $x_1x_2 = 0$ it follows that $x = \pm e_i$ for $i = 1$ or 2; from $y_1y_2 \neq 0$ it follows that $y \neq \pm e_j$ for $j = 1$ and 2.

4. If $x_1x_2 = y_1y_2 = 0$, then $x = \pm e_i$ and $y = \pm e_j$ for $i, j = 1$ or 2. Thus T_t is of the form

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \mp t \end{bmatrix}, \quad \begin{bmatrix} 0 & \mp t \\ \pm 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} \mp t & 0 \\ 0 & \pm 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & \pm 1 \\ \mp t & 0 \end{bmatrix}.$$

Therefore, $T_t \in \mathscr{I}_{x,y}$ iff $|t| \leqslant 1$, so $s = 1$ or -1.

Let now $1 < p < 2$. Then T is extreme iff T' is extreme and we have $q = p/(p-1) > 2$. If T attains its norm on a vector z, then T' attains its norm on $(Tz)^1$. Therefore, we obtain the characterization of the extreme contractions also for $1 < p < 2$ and the proof is complete.

Corollary. *For $p \neq 2$ the set of all extreme contractions in the n-dimensional l_p-space $(2 \leqslant n < \infty)$ is not closed.*

Proof. Let $x(n) = \left((1-1/n)^{1/p}, (1/n)^{1/p}, 0, \ldots\right)$. Then for $p > 2$ the sequence of extreme contractions

$$T_n = x(n)\otimes e_1 + \sum_{k>2} e_k\otimes e_k$$

(Lemma 1) converges to

$$T = e_1\otimes e_1 + \sum_{k>2} e_k\otimes e_k$$

and, obviously, T is not extreme. In case of $1 < p < 2$ the proof is analogous.

We recall that a convex compact set Q in a Euclidean space is said to be *stable* if all k-skeletons of Q are closed (see [6]). Therefore, in particular,

($*$) *The unit ball of operators on the n-dimensional real l_p-space ($n \geqslant 2$, $p \neq 2$) is not stable.*

Let E and F be real Hilbert spaces and let T be a contraction from E to F. We write

$$E(T) = \{x \in E\colon \|Tx\| = \|x\|\} \quad \text{and} \quad F(T) = \{y \in F\colon \|T^*y\| = \|y\|\}.$$

If $x \in E(T)$, then

$$\|x\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leqslant \|T^*Tx\|\,\|x\| \leqslant \|x\|^2.$$

Therefore, $T^*Tx = x$ and we obtain

$$\langle Tx, Tz\rangle = \langle T^*Tx, z\rangle = \langle x, z\rangle \quad \text{for each } z \in E.$$

In particular, if $x, z \in E(T)$, then

$$\|T(x+z)\|^2 = \langle T^*Tx + T^*Tz, x+z\rangle = \|x+z\|^2.$$

Therefore, $E(T)$ is a linear subspace of E and we have $T\big(E(T)\big) = F(T)$ and $T\big(E(T)^\perp\big) \subset F(T)^\perp$. Thus, if $E(T)^\perp \neq \{0\} \neq F(T)^\perp$ and $\|T\,|\,E(T)^\perp\|$ < 1 (in particular, if $E(T)^\perp$ is finite dimensional), then T is not an extreme contraction. Indeed, $\|T \pm R\| \leqslant 1$ if we let $R = \varepsilon TP \neq 0$, where $\varepsilon > 0$ is such that $(1+\varepsilon)\|T\,|\,E(T)^\perp\| \leqslant 1$ and P is an orthogonal projection onto $E(T)^\perp$.

Therefore, we obtain the well-known assertion that an operator T on the finite-dimensional Euclidean space is an extreme contraction if and only if T is an isometry (cf. [5]).

Now for each contraction T we define $d(T)$ as the dimension of the face generated by T in the unit ball of $\mathscr{L}(\boldsymbol{R}^n, \boldsymbol{R}^n)$. Since $\|T\,|\,E(T)^\perp\| < 1$, we have

$$d(T) \geqslant \dim \mathscr{L}\big(E(T)^\perp,\ F(T)^\perp\big) = \dim\big(E(T)^\perp\big)\dim\big(F(T)^\perp\big) = \big(\dim E(T)^\perp\big)^2.$$

Since, clearly, $S\,|\,E(T) = T\,|\,E(T)$ whenever S is in the face generated by T, we have $d(T) = \big(\dim E(T)\big)^2$. Now we prove

($**$) *The unit ball of $\mathscr{L}(\boldsymbol{R}^n, \boldsymbol{R}^n)$ is stable.*

Indeed, it is sufficient to show that the mapping $T \to d(T)$ is lower semicontinuous on the unit ball. Let T be a contraction in $\mathscr{L}(\boldsymbol{R}^n, \boldsymbol{R}^n)$. We put $\|T\,|\,E(T)^\perp\| = 1-\varepsilon$. If $d(T) > d(S)$ for a contraction S, then $\dim E(S)^\perp < \dim E(T)^\perp$, and so

$$\dim E(S) + \dim E(T)^\perp > n.$$

Thus there exists a unit vector x in $E(S)\cap E(T)^{\perp}$ and we have

$$\|S-T\| \geqslant \|Sx-Tx\| \geqslant \|Sx\|-\|Tx\| \geqslant 1-(1-\varepsilon) = \varepsilon.$$

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