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ON A THEOREM OF JONES AND HEATH CONCERNING SEPARABLE NORMAL SPACES

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1. Introduction. In 1937 Jones [6] proved that if $2^{\aleph_0} < 2^{\aleph_1}$, then every uncountable subset of a normal separable space has a limit point. In 1964 Heath [5], using a set-theoretic result due to Fichtenholz and Kantorovitch [3], proved the converse of the Jones theorem. The main result of this paper (Section 4) is the extension of the Jones-Heath theorem to higher cardinals. In Section 3 we give a brief but systematic study of the relationship between the density of a space, its separation axioms, and the existence of subsets of large cardinality having no limit point. The techniques and discussion in this section should motivate our proof that Heath's theorem extends to higher cardinals. Hausdorff's generalization [4] of the Fichtenholz and Kantorovitch result (see Section 2) plays a key role in our proofs.

Throughout this paper $\mathfrak{m}$ and $\mathfrak{n}$ are infinite cardinals with $\mathfrak{m} < \mathfrak{n}$, α and β are ordinals, and the cardinality of a set E is denoted by $|E|$. All regular, normal, and collectionwise normal spaces are T_1. The *density* of a space X, denoted by $d(X)$, is $\aleph_0 \cdot \mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal such that X has a dense subset of cardinality $\mathfrak{m}$.

2. Hausdorff's theorem. A collection $\mathscr{A}$ of subsets of a set E is said to be *independent* if, given any finite collection $A_1, \ldots, A_i, B_1, \ldots, B_j$ of distinct elements of $\mathscr{A}$,

$$A_1 \cap \ldots \cap A_i \cap (E - B_1) \cap \ldots \cap (E - B_j) \neq \emptyset.$$

The following theorem was proved by Fichtenholz and Kantorovitch [3] for the cases $\aleph_0$ and $2^{\aleph_0}$, and was later extended to all infinite cardinals by Hausdorff [4]:

THEOREM 2.1 (Hausdorff). *Let $\mathfrak{m}$ be an infinite cardinal, and let E be a set with $|E| = \mathfrak{m}$. Then there is an independent collection $\mathscr{A}$ of subsets of E with $|\mathscr{A}| = 2^{\mathfrak{m}}$.*

The following are easy consequences of Hausdorff's theorem. Corollary 2.1 is well known. For the sake of completeness, we sketch the proof of Corollary 2.2. (Recall that an ultrafilter $\mathscr{U}$ on a set E is *free*, or *non-principal*, if $\bigcap \mathscr{U} = \emptyset$.)

COROLLARY 2.1. *Let $\mathfrak{m}$ be an infinite cardinal, and let E be a set with $|E| = \mathfrak{m}$. Then there are $2^{2^{\mathfrak{m}}}$ free ultrafilters on E.*

COROLLARY 2.2. *Let $\mathfrak{m}$ be an infinite cardinal, and let E be a set with $|E| = \mathfrak{m}$. Then there is a collection of $2^{\mathfrak{m}}$ free ultrafilters on the set E, say $\{\mathscr{U}_a\colon 0 \leqslant a < 2^{\mathfrak{m}}\}$, such that, given any $a < 2^{\mathfrak{m}}$, there is a subset U of E such that $U \in \mathscr{U}_a$ and $(E - U) \in \mathscr{U}_\beta$ for all $\beta < 2^{\mathfrak{m}}$, $\beta \neq a$.*

Proof. Let $\mathscr{A} = \{A_a\colon 0 \leqslant a < 2^{\mathfrak{m}}\}$ be an independent collection of distinct subsets of E. For each $a < 2^{\mathfrak{m}}$ let

$$\mathscr{F}_a = \{A_a\} \cup \{E - A_\beta\colon 0 \leqslant \beta < 2^{\mathfrak{m}}, \beta \neq a\}.$$

Clearly, $\mathscr{F}_a$ is a filter subbase on E, and hence is included in an ultrafilter $\mathscr{U}_a$. Now consider the collection $\{\mathscr{U}_a\colon 0 \leqslant a < 2^{\mathfrak{m}}\}$. It is clear that $\mathscr{U}_a \neq \mathscr{U}_\beta$ whenever a and β are distinct elements of $2^{\mathfrak{m}}$, and that, given any $a < 2^{\mathfrak{m}}$, $A_a \in \mathscr{U}_a$ and $(E - A_a) \in \mathscr{U}_\beta$ for all $\beta \neq a$, $\beta < 2^{\mathfrak{m}}$. Since at most $\mathfrak{m}$ of these ultrafilters are not free, the proof is complete.

3. A general topological problem. Let $\mathfrak{m}$ and $\mathfrak{n}$ be infinite cardinals with $\mathfrak{m} < \mathfrak{n}$. Is there a topological space X with $d(X) = \mathfrak{m}$ having a subset of cardinality $\mathfrak{n}$ with no limit point? As we now show, the answer (i.e., the gap between $\mathfrak{m}$ and $\mathfrak{n}$) depends upon the separation axioms which X satisfies.

If we require that X be just T_1, then the gap between $\mathfrak{m}$ and $\mathfrak{n}$ can be arbitrarily large. Indeed, given infinite cardinals $\mathfrak{m}$ and $\mathfrak{n}$ with $\mathfrak{m} < \mathfrak{n}$, let E and F be disjoint sets with $|E| = \mathfrak{m}$ and $|F| = \mathfrak{n}$, and let $X = E \cup F$. The collection

$$\mathscr{B} = \{\{p\}\colon p \in E\} \cup \{\{q\} \cup U\colon q \in F,\ U \subseteq E,\ E - U \text{ finite}\}$$

is a base for a T_1-topology on X. Clearly, E is a dense subset of X of cardinality $\mathfrak{m}$, and F is a subset of X of cardinality $\mathfrak{n}$ having no limit point.

Now suppose X is a Hausdorff space with $d(X) = \mathfrak{m}$. By a well-known result (see [7], p. 10), $|X| \leqslant 2^{2^{d(X)}} = 2^{2^{\mathfrak{m}}}$, and so X cannot have a subset of cardinality greater than $2^{2^{\mathfrak{m}}}$ having no limit point. On the other hand, let E be a set with $|E| = \mathfrak{m}$, let $\{\mathscr{U}_a\colon 0 \leqslant a < 2^{2^{\mathfrak{m}}}\}$ be a collection of $2^{2^{\mathfrak{m}}}$ distinct free ultrafilters on E (use Corollary 2.1), let $X = E \cup 2^{2^{\mathfrak{m}}}$, and assume $E \cap 2^{2^{\mathfrak{m}}} = \emptyset$. The collection

$$\mathscr{B} = \{\{p\}\colon p \in E\} \cup \{\{a\} \cup U\colon 0 \leqslant a < 2^{2^{\mathfrak{m}}},\ U \in \mathscr{U}_a\}$$

is a base for a Hausdorff topology on X. Moreover, E is a dense subset of X of cardinality $\mathfrak{m}$, and $2^{2^{\mathfrak{m}}}$ is a subset of X of cardinality $2^{2^{\mathfrak{m}}}$ having no limit point. In summary, we have the following result:

THEOREM 3.1. *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be infinite cardinals with* $\mathfrak{m} < \mathfrak{n}$. *Then there is a Hausdorff space* X *with* $d(X) = \mathfrak{m}$ *having a subset of cardinality* $\mathfrak{n}$ *with no limit point if and only if* $\mathfrak{n} \leqslant 2^{2^{\mathfrak{m}}}$.

Now let X be a regular space with $d(X) = \mathfrak{m}$. Then X has a base of cardinality not greater than $2^{\mathfrak{m}}$ (see [7], p. 10), from which it easily follows that every subset of X of cardinality greater than $2^{\mathfrak{m}}$ has a limit point. Now let E be a set with $|E| = \mathfrak{m}$, let $\{\mathscr{U}_a\colon 0 \leqslant a < 2^{\mathfrak{m}}\}$ be $2^{\mathfrak{m}}$ distinct free ultrafilters on E as in Corollary 2.2, let $X = E \cup 2^{\mathfrak{m}}$, and assume $E \cap 2^{\mathfrak{m}} = \emptyset$. The collection

$$\mathscr{B} = \big\{\{p\}\colon p \in E\big\} \cup \big\{\{a\} \cup U\colon 0 \leqslant a < 2^{\mathfrak{m}},\ U \in \mathscr{U}_a\big\}$$

is a base for a regular topology on X. Moreover, E is a dense subset of X with $|E| = \mathfrak{m}$ and $2^{\mathfrak{m}}$ is a subset of X of cardinality $2^{\mathfrak{m}}$ having no limit point. This establishes the following result:

THEOREM 3.2. *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be infinite cardinals with* $\mathfrak{m} < \mathfrak{n}$. *Then there is a regular space* X *with* $d(X) = \mathfrak{m}$ *having a subset of cardinality* $\mathfrak{n}$ *with no limit point if and only if* $\mathfrak{n} \leqslant 2^{\mathfrak{m}}$.

The situation for normal spaces provides the most intriguing case, and is discussed in the next section. Finally, note that, for any collectionwise normal space X with $d(X) = \mathfrak{m}$, every subset of cardinality greater than $\mathfrak{m}$ has a limit point.

Remarks. Theorems 3.1 and 3.2 were announced in [10]. The example constructed in Theorem 3.1 is the Katětov extension $\varkappa E$ [8] of a discrete space E with $|E| = \mathfrak{m}$. The example in Theorem 3.2 can be constructed using Corollary 2.2 and the Čech-Stone compactification of a discrete space E with $|E| = \mathfrak{m}$. See [2], p. 133, for the case $\mathfrak{m} = \aleph_0$.

4. The main theorem. In this section we extend the theorem of Jones and Heath to higher cardinality. We begin with a proposition which is suggested by Corollary 2.2 and the constructions in Section 3:

PROPOSITION 4.1. *Let* $\mathfrak{m}$ *be an infinite cardinal, and let* E *be a set with* $|E| = \mathfrak{m}$. *Then there is a collection of* $\mathfrak{m}$ *free ultrafilters on* E, *say* $\{\mathscr{U}_a\colon 0 \leqslant a < \mathfrak{m}\}$, *such that, given any subset* D *of* $\mathfrak{m}$, *there is some* $U \subseteq E$ *such that* $U \in \mathscr{U}_a$ *for all* a *in* D *and* $(E - U) \in \mathscr{U}_a$ *for all* a *in* $\mathfrak{m} - D$.

Proof. Clearly, it suffices to construct the ultrafilters on $\mathfrak{m} \times \mathfrak{m}$. For a, β in $\mathfrak{m}$ let

$$F_{a\beta} = \{(a, \gamma)\colon \beta \leqslant \gamma < \mathfrak{m}\},$$

and for each a in $\mathfrak{m}$ let

$$\mathscr{F}_a = \{F_{a\beta}\colon 0 \leqslant \beta < \mathfrak{m}\}.$$

Clearly, $\mathscr{F}_a$ is a filter base on $\mathfrak{m} \times \mathfrak{m}$, and hence is included in an ultrafilter $\mathscr{U}_a$. Consider the collection $\{\mathscr{U}_a: 0 \leqslant a < \mathfrak{m}\}$. It is easy to check that each $\mathscr{U}_a$ is free and that $\mathscr{U}_a \neq \mathscr{U}_\beta$ whenever $a \neq \beta$. Finally, given $D \subseteq \mathfrak{m}$, the set $U = \bigcup\{\{a\} \times \mathfrak{m}: a \in D\}$ belongs to $\mathscr{U}_a$ if and only if $a \in D$.

It is natural to ask if there is a collection of more than $\mathfrak{m}$ ultrafilters on the set E having the property stated in Proposition 4.1. The existence of such a collection is a key step in the extension of the Jones-Heath theorem to higher cardinals.

THEOREM 4.1. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be infinite cardinals with $\mathfrak{m} < \mathfrak{n}$. The following are equivalent:*

(1) *There is a normal space X with $d(X) = \mathfrak{m}$ having a subset of cardinality $\mathfrak{n}$ with no limit point.*

(2) $2^{\mathfrak{n}} = 2^{\mathfrak{m}}$.

(3) *Given any set E with $|E| = \mathfrak{m}$, there is a collection of $\mathfrak{n}$ free ultrafilters on E, say $\{\mathscr{U}_a: 0 \leqslant a < \mathfrak{n}\}$, such that, given any $D \subseteq \mathfrak{n}$, there is some $U \subseteq E$ such that $U \in \mathscr{U}_a$ for all a in D and $(E - U) \in \mathscr{U}_a$ for all a in $\mathfrak{n} - D$.*

Proof. The proof of (1) $\Rightarrow$ (2) is an obvious modification of the Jones proof in [6], and so is omitted.

(2) $\Rightarrow$ (3). Given the set E with $|E| = \mathfrak{m}$, let $\mathscr{A}$ be an independent collection of $2^{\mathfrak{m}}$ subsets of E. Let $\mathscr{B} = \{E - A: A \in \mathscr{A}\}$, and note that $\mathscr{A} \cap \mathscr{B} = \emptyset$. We now construct a one-one function f from $\mathscr{P}(\mathfrak{n})$ onto $\mathscr{A} \cup \mathscr{B}$ such that $f(\mathfrak{n} - D) = E - f(D)$ for each $D \subseteq \mathfrak{n}$. Let $\mathscr{C}$ be a maximal subcollection of $\mathscr{P}(\mathfrak{n})$ such that $C \in \mathscr{C}$ implies $\mathfrak{n} - C \notin \mathscr{C}$. The maximality of $\mathscr{C}$ implies that $|\mathscr{C}| = 2^{\mathfrak{n}}$ and, for each subset D of $\mathfrak{n}$, $D \in \mathscr{C}$ or $(\mathfrak{n} - D) \in \mathscr{C}$. Since $2^{\mathfrak{m}} = 2^{\mathfrak{n}}$, there is a one-one function g from $\mathscr{C}$ onto $\mathscr{A}$. Now let $D \subseteq \mathfrak{n}$, and define $f(D)$ to be $g(D)$ if $D \in \mathscr{C}$; otherwise, define $f(D)$ to be $E - g(\mathfrak{n} - D)$. Clearly, f is the desired function. Note that f has the following properties:

(a) $f(D) \cap f(\mathfrak{n} - D) = \emptyset$ for each $D \subseteq \mathfrak{n}$.

(b) If $D_1, \ldots, D_k$ are distinct subsets of $\mathfrak{n}$ such that $D_i \neq \mathfrak{n} - D_j$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k$, then

$$\bigcap_{i=1}^{k} f(D_i) \neq \emptyset.$$

For each $a < \mathfrak{n}$ let $\mathscr{F}_a = \{f(D): a \in D \subseteq \mathfrak{n}\}$. From (b) it follows that $\mathscr{F}_a$ is a filter subbase on E, and hence is included in an ultrafilter $\mathscr{U}_a$. Now consider the collection $\{\mathscr{U}_a: 0 \leqslant a < \mathfrak{n}\}$. From (a) it follows that $\mathscr{U}_a \neq \mathscr{U}_\beta$ whenever a and β are distinct elements of $\mathfrak{n}$. Moreover, given $D \subseteq \mathfrak{n}$, $f(D)$ is a subset of E which belongs to $\mathscr{U}_a$ if and only if $a \in D$. Since at most $\mathfrak{m}$ of these ultrafilters are not free, and $\mathfrak{m} < \mathfrak{n}$, the proof is complete.

(3) $\Rightarrow$ (1). Let E be a set with $|E| = \mathfrak{m}$, let $\{\mathscr{U}_a: 0 \leqslant a < \mathfrak{n}\}$ be a collection of $\mathfrak{n}$ free ultrafilters on E as in (3), let $X = E \cup \mathfrak{n}$, and assume

$E \cap \mathfrak{n} = \emptyset$. The collection

$$\mathscr{B} = \{\{p\}: p \in E\} \cup \{\{a\} \cup U: 0 \leqslant a < \mathfrak{n}, U \in \mathscr{U}_a\}$$

is a base for a normal topology on X. Moreover, E is a dense subset of X of cardinality $\mathfrak{m}$ and $\mathfrak{n}$ is a subset of X of cardinality $\mathfrak{n}$ having no limit point.

Remarks. See [9] for a somewhat different proof of (2) $\Rightarrow$ (1). The results for regularity, normality, and collectionwise normality remain true if the T_1-hypothesis is omitted. The normality of Theorem 4.1 (1) can be strengthened to hereditary normality; in addition, one may then replace "subset of cardinality $\mathfrak{n}$ with no limit point" by "discrete subspace of cardinality $\mathfrak{n}$". The proofs remain essentially the same.

Theorem 4.1 takes on added interest in view of [1], where Easton shows that, roughly speaking, any behavior of the exponent function for regular cardinals which is not obviously refutable is consistent with axioms of set theory. Thus one can have (the topological translation of) $2^{\aleph_0} = 2^{\aleph_1} < 2^{\aleph_2} = 2^{\aleph_3}$.

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