COLLOQUIUM MATHEMATICUM

LI DÉDIÉ À M. STANISŁAW HARTMAN 1987

A NUMERICAL CONSTANT ASSOCIATED WITH GENERALIZED CONVOLUTIONS

BY

K. URBANIK (WROCŁAW)

The main topic of this paper is a description of generalized convolutions in terms of some invariance properties of their characteristic functions. We denote by C_b the space of bounded continuous real-valued functions on the positive half-line R^+ with the topology of uniform convergence on every compact subset of R^+. Further, by $\mathfrak{P}$ we shall denote the set of all probability measures defined on Borel subsets of R^+. The set $\mathfrak{P}$ is endowed with the topology of weak convergence. For $a \in R^+$ we define the mapping T_a: $R^+ \to R^+$ by $T_a x = ax$. For a function $f \in C_b$, $T_a f$ denotes the function $(T_a f)(x) = f(ax)$ and for a measure $\mu \in \mathfrak{P}$, $T_a \mu$ denotes the measure defined by $(T_a \mu)(E) = \mu(a^{-1} E)$ if $a > 0$ and $T_0 \mu = \delta_0$, where for $a \in R^+$, $B \subset R^+$ we put $aB = \{ab\colon b \in B\}$ and δ_c is the probability measure concentrated at the point c. We shall also use the notation $AB = \{ab\colon a \in A, b \in B\}$ for $A, B \subset R^+$. We say that two functions f and g from C_b are *similar* if $f = T_a g$ for a certain positive number a.

A continuous commutative and associative $\mathfrak{P}$-valued binary operation $\circ$ defined on $\mathfrak{P}$ is called a *generalized convolution* if the following conditions are fulfilled:

(i) the measure δ_0 is a unit element, i.e. $\mu \circ \delta_0 = \mu$ $(\mu \in \mathfrak{P})$,

(ii) $(c\mu + (1-c)\nu) \circ \lambda = c(\mu \circ \lambda) + (1-c)(\nu \circ \lambda)$ $(0 \leqslant c \leqslant 1,\ \mu, \nu, \lambda \in \mathfrak{P})$,

(iii) $(T_a \mu) \circ (T_a \nu) = T_a(\mu \circ \nu)$ $(a \in R^+,\ \mu, \nu \in \mathfrak{P})$,

(iv) there exists a sequence $c_1, c_2, \ldots$ of positive numbers such that the sequence $T_{c_n} \delta_1^{\circ n}$ converges to a measure different from δ_0.

The power $\delta_1^{\circ n}$ is taken here in the sense of the operation $\circ$.

The set $\mathfrak{P}$ with the operation $\circ$ and the operations of convex linear combinations is called a *generalized convolution algebra* and denoted by $(\mathfrak{P}, \circ)$. For basic properties of generalized convolution algebras we refer to [1] and [3]–[13]. In particular, generalized convolution algebras admitting a non-trivial homomorphism into the algebra of real numbers with the operations of multiplication and convex linear combinations are called *regular*. We recall that a homomorphism h is *trivial* if either $h \equiv 0$ or $h \equiv 1$. All generalized convolution algebras under consideration in the sequel will tacitly be assumed to be regular. It has been proved in [10] (Theorem 6) that each regular generalized convolution algebra admits a characteristic function, i.e. a homeomorphic map from $\mathfrak{P}$ into C_b: $\mu \to \hat{\mu}$ such that $(c\mu+(1-c)\nu)^{\wedge} = c\hat{\mu}+(1-c)\hat{\nu}$ $(0 \leqslant c \leqslant 1)$, $(\mu \circ \nu)^{\wedge} = \hat{\mu}\hat{\nu}$ and $(T_a \mu)^{\wedge} = T_a \hat{\mu}$ $(a \in R^+)$ for all $\mu, \nu \in \mathfrak{P}$. The characteristic function plays the same fundamental role in generalized convolution algebras as the Laplace transform in the ordinary convolution algebra. Moreover, each characteristic function is an integral transform

$$\hat{\mu}(t) = \int_0^\infty \Omega(tx)\,\mu(dx)$$

with a continuous kernel Ω satisfying the conditions $|\Omega(t)| \leqslant 1$ $(t \in R^+)$ and

$$\Omega(t) = 1 - t^{\varkappa} L(t), \tag{1}$$

where $\varkappa > 0$ and the function L is slowly varying and continuous at the origin. By Theorem 2.1 in [12] all kernels corresponding to characteristic functions of a generalized convolution algebra are similar. Consequently, the constant $\varkappa$ in (1) does not depend upon the choice of a characteristic function and is called the *characteristic exponent* of the generalized convolution $\circ$, in symbols $\varkappa(\circ) = \varkappa$. Further, by $*_{\alpha,1}$ $(\alpha > 0)$ we shall denote the Kingman convolution defined by the formula

$$\int_0^\infty f(x)(\mu *_{\alpha,1} \nu)(dx) = \tfrac{1}{2}\int_0^\infty \int_0^\infty \left[f\left((x^\alpha+y^\alpha)^{1/\alpha}\right)+f\left(|x^\alpha-y^\alpha|^{1/\alpha}\right)\right]\mu(dx)\,\nu(dy)$$

for all $f \in C_b$. In this case the kernel Ω is given by the formula

$$\Omega(t) = \cos t^\alpha \tag{2}$$

and

$$\varkappa(*_{\alpha,1}) = 2\alpha. \tag{3}$$

For any pair μ, ν from $\mathfrak{P}$ by $\mu\nu$ we shall denote the probability distribution of the product XY of two independent random variables X and Y with probability distributions μ and ν respectively. By Proposition 1.3 in [12] for every characteristic function $\mu \to \hat{\mu}$ of a generalized convolution the formula

$$(\mu\nu)^{\wedge}(t) = \int_0^\infty \hat{\mu}(tx)\,\nu(dx) \tag{4}$$

is true. Further, by Theorem 7 in [10] for each characteristic function of a generalized convolution $\circ$ with $\varkappa(\circ) = \varkappa$ there exists a probability measure $\sigma_\varkappa$ called the *characteristic measure* of $\circ$ such that

$$\hat{\sigma}_\varkappa(t) = \exp(-t^\varkappa) \quad (t \in R^+). \tag{5}$$

By Theorems 5 and 6 in [10] for any kernel Ω there exists a positive number t_0 such that $\Omega(t) < 1$ whenever $0 < t < t_0$. Consequently, without loss of generality passing to similar kernels if necessary we may assume that the kernel in question has one of the following properties:

$$(*) \qquad \Omega(1) = 1, \quad \Omega(t) < 1 \quad \text{whenever } 0 < t < 1,$$

$$(**) \qquad \Omega(t) < 1 \quad \text{for all } t > 0.$$

Consider two generalized convolutions $\circ$ and $\circ'$. The convolution $\circ$ is said to be *representable in* $\circ'$, in symbols $\circ \prec \circ'$, if there exists a continuous non-trivial homomorphism from the algebra $(\mathfrak{P}, \circ)$ into the algebra $(\mathfrak{P}, \circ')$ commuting with the semigroup T_a $(a \in R^+)$ ([12], Chapter 3). We recall that a homomorphism h is *non-trivial* if $h(\mu) \not\equiv \delta_0$ for $\mu \in \mathfrak{P}$.

For any measure μ from $\mathfrak{P}$ by $S(\mu)$ we shall denote its support. Given a generalized convolution $\circ$ we put

$$A(\mu) = \{t\colon \hat{\mu}(t) = 1,\, t \in R^+\}$$

where $\mu \to \hat{\mu}$ is the characteristic function of $\circ$. Of course, we may assume that its kernel fulfils one of the conditions $(*)$, $(**)$. It is clear that the set $A(\mu)$ is closed,

$$0 \in A(\mu), \tag{6}$$

$$A(\mu \circ \nu) \supset A(\mu) \cap A(\nu). \tag{7}$$

Moreover, by formula (4),

$$S(\nu) A(\mu\nu) \subset A(\mu). \tag{8}$$

Hence we get the inclusion

$$A(\mu\nu) \subset \bigcap_{x \in S(\nu)\setminus\{0\}} x^{-1} A(\mu)$$

if $\nu \neq \delta_0$. The converse inclusion is also true. Namely, if $\nu \neq \delta_0$ and $t \in x^{-1} A(\mu)$ for every $x \in S(\nu)\setminus\{0\}$, then, by (6), $tx \in A(\mu)$ for every $x \in S(\nu)$ which, by (4), yields $t \in A(\mu\nu)$. Thus

$$A(\mu\nu) = \bigcap_{x \in S(\nu)\setminus\{0\}} x^{-1} A(\mu) \quad \text{if } \nu \neq \delta_0. \tag{9}$$

Setting $\nu = \delta_1 \circ \delta_1$ and $\mu = \delta_1$ into (8) and applying (7) we get the inclusion

$$S(\delta_1 \circ \delta_1) A(\delta_1) \subset A(\delta_1). \tag{10}$$

Further, it is easy to see that

$$1 \in A(\delta_1), \quad A(\delta_1) \cap (0, 1) = \emptyset \quad \text{in the case } (*), \tag{11}$$

and

$$A(\mu) = \{0\} \quad (\mu \in \mathfrak{P}) \quad \text{in the case } (**). \tag{12}$$

LEMMA 1. *In the case* $(*)$ *we have the relation* $S(\delta_1 \circ \delta_1) \cap (1, \infty) \neq \emptyset$.

Proof. Suppose the contrary, i.e. $S(\delta_1 \circ \delta_1) \subset [0, 1]$. Then, by (10) and (11), $S(\delta_1 \circ \delta_1) \subset \{0\} \cup \{1\}$ or, in other words, $\delta_1 \circ \delta_1 = c\delta_0 + (1-c)\delta_1$ where $0 \leqslant c \leqslant 1$. Passing to the characteristic functions we have the equation $\Omega^2(t) = c + (1-c)\Omega(t)$ which shows that Ω assumes at most two values 1 and $-c$. Since $\Omega(0) = 1$, by continuity of Ω, we get $\Omega \equiv 1$. But this gives the contradiction. The Lemma is thus proved.

Invariance properties of the set $A(\delta_1)$ are described by the multiplicative semigroup

$$N = \{a\colon\ T_a A(\delta_1) \subset A(\delta_1),\ a > 1\}.$$

Setting $\mu = \delta_1$ into (9) we obtain the formula

$$A(\nu) = \bigcap_{x \in S(\nu) \setminus \{0\}} x^{-1} A(\delta_1)$$

if $\nu \neq \delta_0$. As an immediate consequence of the above formula and the equation $A(\delta_0) = R^+$ we get the inclusion

$$NA(\nu) \subset A(\nu) \quad (\nu \in \mathfrak{P}). \tag{13}$$

In particular,

$$N \subset A(\nu) \quad \text{if } \hat{\nu}(1) = 1. \tag{14}$$

Further, by (12),

$$N = (1, \infty) \quad \text{in the case } (**) \tag{15}$$

and, by (10),

$$S(\delta_1 \circ \delta_1) \cap (1, \infty) \subset N \quad \text{in the case } (*) \tag{16}$$

which, by Lemma 1, shows that always $N \neq \emptyset$. Moreover, in the case $(*)$ the semigroup N is closed, i.e. 1 does not belong to the closure of N. In fact, the contrary would imply $N = (1, \infty)$ and, by (14), $(1, \infty) \subset A(\delta_1)$. In other words, $\Omega(t) = \Omega(0) = 1$ for $t > 1$. But then the left-hand side of the equation

$$\int_0^\infty \Omega(tx)\sigma_\varkappa(dx) = \exp(-t^\varkappa),$$

where $\sigma_\varkappa$ is the characteristic measure of the convolution in question, would tend to 1 when $t \to \infty$. This contradiction shows that in the case $(*)$ the semigroup N is closed.

We associate with every generalized convolution $\circ$ a numerical constant by setting

$$\eta(\circ) = \inf N.$$

Of course

(17) $$\eta(\circ) > 1 \quad \text{in the case } (*)$$

and, by (15),

(18) $$\eta(\circ) = 1 \quad \text{in the case } (**).$$

We note that for the convolution $*_{\alpha,1}$ $N = \{n^{1/\alpha}\colon n \geqslant 2\}$ and

(19) $$\eta(*_{\alpha,1}) = 2^{1/\alpha}.$$

THEOREM 1. *If* $\circ \prec \circ'$, *then* $\eta(\circ) \leqslant \eta(\circ')$.

Proof. Suppose that h is a non-trivial continuous homomorphism from $(\mathfrak{P}, \circ)$ into $(\mathfrak{P}, \circ')$ commuting with the semigroup T_a $(a \in R^+)$. By Lemma 3.1 in [12] the map h is of the form $h(\mu) = \lambda\mu$ for a certain non-degenerate $\lambda \in \mathfrak{P}$. Moreover, if $\mu \to \hat{\mu}'$ is a characteristic function of $\circ'$, then $\mu \to [(h(\mu))^{\wedge}]'$ is a characteristic function of $\circ$. Therefore denoting by $A(\mu)$ and $A'(\mu)$ the sets for the convolutions $\circ$ and $\circ'$ respectively we may assume without loss of generality that

$$A(\mu) = A'(\lambda\mu) \quad (\mu \in \mathfrak{P}).$$

Let N' be the invariance semigroup for $\circ'$. The last equation and (13) imply the inclusion $N' A(\delta_1) \subset A(\delta_1)$. Thus $N' \subset N$ and, consequently, $\eta(\circ) \leqslant \eta(\circ')$ which completes the proof.

LEMMA 2. *For every generalized convolution* $\circ$ *with* $\varkappa(\circ) = \varkappa$ *the inequality*

$$\int_0^\infty x^\varkappa (\delta_1 \circ \delta_1)(dx) \leqslant 2$$

is true.

Proof. The formula

$$\Omega^2(t) = \int_0^\infty \Omega(tx)(\delta_1 \circ \delta_1)(dx)$$

implies

$$1 + \Omega(t) = \int_0^\infty \frac{1 - \Omega(tx)}{1 - \Omega(t)} (\delta_1 \circ \delta_1)(dx)$$

whence, by (1) and the Fatou lemma when $t \to 0+$ our assertion follows.

LEMMA 3. *For every generalized convolution* $\circ$ *with the property* $(*)$ *we have the formula*

$$\delta_1 \circ \delta_1 = p\delta_0 + q\delta_1 + (1 - p - q)\nu$$

where $\nu \in \mathfrak{P}$, $S(\nu) \subset [\eta(0), \infty)$, $p, q \geqslant 0$ *and*

$$4p+3q \leqslant 2. \tag{20}$$

Proof. By (10) and (11) we have the inclusion $S(\delta_1 \circ \delta_1) \cap [0, 1] \subset \{0\} \cup \{1\}$ which together with (16) yields the inclusion

$$S(\delta_1 \circ \delta_1) \subset \{0\} \cup \{1\} \cup [\eta(0), \infty).$$

Consequently, the measure $\delta_1 \circ \delta_1$ can be written in the form

$$\delta_1 \circ \delta_1 = p\delta_0 + q\delta_1 + (1-p-q)\nu \tag{21}$$

where $\nu \in \mathfrak{P}$, $S(\nu) \subset [\eta(0), \infty)$ and $p, q \geqslant 0$. It remains to prove inequality (20). For every number c $(0 \leqslant c \leqslant 1)$ let I_c denote the set of all pairs of non-negative real numbers (x, y) satisfying the conditions

$$y^2+4cy+4(1+c)x-4c \leqslant 0, \quad y \leqslant 1-c.$$

Put $a_c = \min(2(\sqrt{c^2+c}-c), 1-c)$ and

$$\varphi_c(y) = \frac{4c-y^2-4cy}{4(1+c)}.$$

It is clear that the set I_c is convex and closed. Moreover, its boundary is the union of the sets

$$\begin{gathered} \{(x, y)\colon\ x = \varphi_c(y),\ 0 \leqslant y \leqslant a_c\}, \\ \{(0, y)\colon\ 0 \leqslant y \leqslant a_c\} \quad \text{and} \quad \left\{(x, 0)\colon\ 0 \leqslant x \leqslant \frac{c}{1+c}\right\}. \end{gathered} \tag{22}$$

Hence it follows that the maximum $M_c = \max\{4x+3y\colon (x, y) \in I_c\}$ is attained on the curve (22). Since the function $4\varphi_c(y)+3y$ is monotone non-decreasing in the interval $0 \leqslant y \leqslant a_c$ we finally get the formula $M_c = 4\varphi_c(a_c)+3a_c$ which by a simple computation yields $M_c = 6(\sqrt{c^2+c}-c)$ if $0 \leqslant c \leqslant \frac{1}{3}$ and $M_c = 2$ if $\frac{1}{3} \leqslant c \leqslant 1$. Consequently, $M_c \leqslant 2$ if $0 \leqslant c \leqslant 1$ and to prove inequality (20) it suffices to show that $(p, q) \in I_c$ for a certain c $(0 \leqslant c \leqslant 1)$. Put

$$-b = \inf\{\Omega(t)\colon\ t \in R^+\}.$$

Of course, $b \leqslant 1$ and $\hat{\mu}(t) \geqslant -b$ $(t \in R^+, \mu \in \mathfrak{P})$ which, by (5), yields $b \geqslant 0$. Further, by (21),

$$\Omega^2(t) = p+q\Omega(t)+(1-p-q)\int_0^\infty \Omega(tx)\,\nu(dx). \tag{23}$$

Since $1 \geqslant q \geqslant 0 \geqslant -b$ we can find, by the continuity of Ω, a sequence $t_1, t_2, \ldots$ of positive real numbers with the property $\Omega(t_n) \to q/2$. Setting

$t = t_n$ into (23) we get, when $n \to \infty$

$$\frac{q^2}{4} \geqslant p + \frac{q^2}{2} - (1-p-q)b$$

or, equivalently,

$$q^2 + 4bq + 4(1+b)p - 4b \leqslant 0.$$

Further, taking a sequence $u_1, u_2, \ldots$ with the property $\Omega(u_n) \to -b$ and setting $t = u_n$ into (23) we get, when $n \to \infty$, $b^2 \leqslant p - bq + 1 - p - q$ or, equivalently, $q \leqslant 1-b$. This shows that $(p, q) \in I_b$ which completes the proof of the Lemma.

A relation between constants $\varkappa(\circ)$ and $\eta(\circ)$ is given by the following quite surprising Theorem.

THEOREM 2. *For every generalized convolution* $\circ$ *the inequality*

$$\eta(\circ)^{\varkappa(\circ)} \leqslant 4 \tag{24}$$

is true. The equation

$$\eta(\circ)^{\varkappa(\circ)} = 4 \tag{25}$$

holds if and only if $\circ = *_{\alpha,1}$ *where* $\alpha = \frac{1}{2}\varkappa(\circ)$.

Proof. By (18) inequality (24) is obvious in the case (**). Consider the case (*). Then, by Lemma 3,

$$\delta_1 \circ \delta_1 = p\delta_0 + q\delta_1 + (1-p-q)\nu \tag{26}$$

where $p, q \geqslant 0$,

$$4p + 3q \leqslant 2, \tag{27}$$

$\nu \in \mathfrak{P}$ and $S(\nu) \subset [\eta(\circ), \infty)$. The last inclusion yields the inequality

$$\int_0^\infty x^{\varkappa(\circ)}\nu(dx) \geqslant \eta(\circ)^{\varkappa(\circ)}. \tag{28}$$

Applying Lemma 2 we have

$$q + (1-p-q)\int_0^\infty x^{\varkappa(\circ)}\nu(dx) \leqslant 2 \tag{29}$$

which, by (28), implies the inequality

$$(1-p-q)\eta(\circ)^{\varkappa(\circ)} \leqslant 2-q. \tag{30}$$

But, by (27), $4(1-p-q) \geqslant 2-q > 1$ which together with (30) gives (24). Further, by (3) and (19), equation (25) holds for $\circ = *_{\alpha,1}$. Suppose now that for a generalized convolution $\circ$ equation (25) is true. Of course, in this case $\circ$ has property (*) and, by (30), $4(1-p-q) \leqslant 2-q$ which together with (27) yields $4p+3q=2$. Thus

$$p=\frac{2-3q}{4}. \tag{31}$$

Further, by (28), $\int_0^\infty x^{\varkappa(\text{o})}\,\nu(dx)\geqslant 4$. Setting (31) into (29) we obtain the converse inequality $\int_0^\infty x^{\varkappa(\text{o})}\,\nu(dx)\leqslant 4$. Thus $\int_0^\infty x^{\varkappa(\text{o})}\,\nu(dx)=4$. Since $S(\nu)\subset[\eta(\text{o}),\infty)$ the last equation and (25) show that the measure ν is concentrated at the point $\eta(\text{o})$. Consequently, by (31), formula (26) can be rewritten in the form

$$\delta_1\circ\delta_1=\frac{2-3q}{4}\delta_0+q\delta_1+\frac{2-q}{4}\delta_{\eta(\text{o})}$$

or, equivalently, in terms of the characteristic functions

$$\Omega^2(t)=\frac{2-3q}{4}+q\Omega(t)+\frac{2-q}{4}\Omega(\eta(\text{o})t). \tag{32}$$

Put

$$F(t)=\frac{2\Omega(t^{1/\alpha})-q}{2-q}\qquad (t\in R^+) \tag{33}$$

where $\alpha=\frac{1}{2}\varkappa(\text{o})$. By (25) and (32) the function F fulfils the equation

$$F^2(t)=\tfrac{1}{2}+\tfrac{1}{2}F(2t).$$

Moreover, the function F is continuous on R^+, $F(0)=1$, $F(t)\leqslant 1$ $(t\in R^+)$, F is not identically equal to 1 and, by (1) and the continuity of the function L, the limit

$$\lim_{t\to 0+}\frac{1-F(t)}{t^2}$$

exists. Applying the Forder theorem ([2], p. 216) we infer that $F(t)=\cos ct$ for a certain positive constant c. Consequently, by (33),

$$\Omega(t)=\left(1-\frac{q}{2}\right)\cos ct^\alpha+\frac{q}{2}. \tag{34}$$

Put $\mu=\frac{2}{2+q}\delta_1\circ\delta_{2^{1/\alpha}}+\frac{q}{2+q}\delta_0$. Then

$$\mu(\{0\})\geqslant\frac{q}{2+q} \tag{35}$$

and, by (34),

$$\hat{\mu}(t) = \frac{2-q}{2(2+q)}\Omega(3^{1/\alpha}t) + \frac{1}{2}\Omega(t) + \frac{q}{2+q}\Omega(2^{1/\alpha}t)$$

or, equivalently,

$$\mu = \frac{2-q}{2(2+q)}\delta_{3^{1/\alpha}} + \frac{1}{2}\delta_1 + \frac{q}{2+q}\delta_{2^{1/\alpha}}.$$

Thus $0 \notin S(\mu)$ which, according to (35), yields $q = 0$. By (34) we infer that $\Omega(t) = \cos ct^{\alpha}$ where $c > 0$. In other words, by (2), Ω is the kernel of the characteristic function of $*_{\alpha,1}$. Since the characteristic function determines the generalized convolution we have the equation $\circ = *_{\alpha,1}$ which completes the proof of the Theorem.

COROLLARY 1. *The convolutions* $*_{\alpha,1}$ $(\alpha > 0)$ *are maximal elements under the partial order* $\prec$, *i.e. the relation* $*_{\alpha,1} \prec \circ$ *yields* $\circ = *_{\alpha,1}$.

Proof. Suppose that $*_{\alpha,1} \prec \circ$. Then, by Theorem 1, $\eta(\circ) \geqslant \eta(*_{\alpha,1})$ and, by Theorem 3.1 in [12], $\varkappa(\circ) \geqslant \varkappa(*_{\alpha,1})$. Consequently, $\eta(\circ)^{\varkappa(\circ)} \geqslant 4$. Applying Theorem 2 we infer that $\eta(\circ)^{\varkappa(\circ)} = 4$, $\eta(\circ) = \eta(*_{\alpha,1})$, $\varkappa(\circ) = \varkappa(*_{\alpha,1}) = 2\alpha$ which finally yields $\circ = *_{\alpha,1}$.

We conclude this paper with the following simple remark. For any function $f \in C_b$ by $\mathrm{Conv}(f)$ we denote the closed convex set spanned by $\{T_a f\colon a \in R^+\}$. As a consequence of Theorem 2 we get the following statement.

COROLLARY 2. *Suppose that* $f \in C_b$ *and the following conditions are fulfilled*:

(a) $f(t) = 1 - t^{2\alpha}L(t)$ *where* $\alpha > 0$ *and the function* L *is slowly varying at the origin*,

(b) *there exists a probability measure* λ *for which*

$$\overline{\lim_{t\to\infty}} \int_0^{\infty} f(tx)\lambda(dx) < 1,$$

(c) *the set* $\mathrm{Conv}(f)$ *is closed under pointwise multiplication of functions*,

(d) *there exists a function* $g \in \mathrm{Conv}(f)$ *with the properties* $g(1) = 1$ *and* $g(t) < 1$ *for* $t \in (1, 2^{1/\alpha})$.

Then $f(t) = \cos ct^{\alpha}$ *for a certain positive constant* c.

Proof. By Theorem 2 in [8] conditions (a), (b) and (c) are sufficient for the function f to be the kernel of a characteristic function of a generalized convolution, say $\circ$. Moreover, $\mathrm{Conv}(f) = \{\hat{\mu}\colon \mu \in \mathfrak{P}\}$. In particular $g = \hat{\nu}$ for a certain $\nu \in \mathfrak{P}$. By (d) $1 \in A(\nu)$ and $A(\nu) \cap (1, \infty) \subset [2^{1/\alpha}, \infty)$ which by (14) yields the inclusion $N \subset [2^{1/\alpha}, \infty)$. Consequently, $\eta(\circ) \geqslant 2^{1/\alpha}$. On the other hand, by (a), $\varkappa(\circ) = 2\alpha$. Thus we have the inequality $\eta(\circ)^{\varkappa(\circ)} \geqslant 4$. Applying Theorem 2 we infer that $\circ = *_{\alpha,1}$ and, consequently, the function f is similar to the function (2) which completes the proof.

REFERENCES

[1] N. H. Bingham, *Factorization theory and domains of attraction for generalized convolution algebras*, Proceedings of the London Mathematical Society 23 (1971), p. 16–30.

[2] H. G. Forder, *On a duplication formula*, The Mathematical Gazette 41 (1957), p. 215–217.

[3] R. Jajte, *Quasi-stable measures in generalized convolution algebras*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques 24 (1976), p. 505–511.

[4] –, *Quasi-stable measures in generalized convolution algebras II*, ibidem 25 (1977), p. 67–72.

[5] Z. J. Jurek, *Some characterization of the class L in generalized convolution algebras*, ibidem 29 (1981), p. 409–415.

[6] –, *Limit distributions in generalized convolution algebras*, Probability and Mathematical Statistics 5 (1985), p. 113–135.

[7] M. Kłosowska, *On the domain of attraction for generalized convolution algebras*, Revue de Mathématiques Pures et Appliquées 22 (1977), p. 669–677.

[8] J. Kucharczak and K. Urbanik, *Quasi-stable functions*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques 22 (1974), p. 263–269.

[9] Nguyen Van Thu, *Multiply self-decomposable measures in generalized convolution algebras*, Studia Mathematica 66 (1979), p. 177–184.

[10] K. Urbanik, *Generalized convolutions*, ibidem 23 (1964), p. 217–245.

[11] –, *Generalized convolutions II*, ibidem 45 (1973), p. 57–70.

[12] –, *Generalized convolutions III*, ibidem 80 (1984), p. 167–189.

[13] V. E. Volkovich, *On an analytical description of Urbanik algebras*, Izvestija Akademii Nauk UzSSR, Serija Fiziko-Matematičeskih Nauk 5 (1973), p. 12–17.

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
WROCŁAW, POLAND

Reçu par la Rédaction le 15.09.1983