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ON FIXED POINT FREE INVOLUTIONS OF $R^1 \times S^2$

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In 1962, Tao [4] showed that if h is a fixed point free involution of $S^1 \times S^2$, then the orbit space $S^1 \times S^2/h$ must be homeomorphic to either $S^1 \times S^2$ or $S^1 \times P^2$ of $P^3 \# P^3$ or the non-orientable 2-sphere bundle over S^1. The question naturally arises as to whether or not similar results can be obtained when the compact factor S^1 is replaced by the non-compact factor R^1. We answer this question affirmatively by showing that if h is a fixed point free involution of $R^1 \times S^2$, then $R^1 \times S^2/h$ must be homeomorphic either to $R^1 \times P^2$ or to the open 3-dimensional Möbius band.

It follows from Chapter 2 of [3] that h may be viewed as a piecewise linear homeomorphism on some triangulation of $R^1 \times S^2$. For this reason we shall consider all objects of this paper from the polyhedral point of view.

The interior of a topological manifold M will be denoted by $\operatorname{int} M$, and the boundary of M by ∂M. The Euclidean n-dimensional space will be denoted by R^n, and the projective n-space by P^n. The n-dimensional sphere and the n-dimensional ball will be denoted by S^n and B^n, respectively. A space homeomorphic to $R^1 \times S^1$ will be called an *open annulus*. The *open n-dimensional Möbius band* is the space obtained from $R^n - \operatorname{int} B^n$ by identifying antipodal points on $\partial(R^n - \operatorname{int} B^n) = S^{n-1}$. We note that the space thus obtained is homeomorphic to the space obtained from P^n by removing an n-ball, since P^n can be obtained from B^n by identifying antipodal points on ∂B^n.

It will be convenient to view $R^1 \times S^2$ as $(-1, 1) \times S^2$.

LEMMA 1. *If h is a fixed point free involution of $R^1 \times S^2$, then there is a 2-sphere S in $R^1 \times S^2$ which is isotopic to $0 \times S^2$ and such that $hS = S$.*

Proof. Let $S = 0 \times S^2$. If $hS \neq S$ and $S \cap hS \neq \emptyset$, then, by using small isotopic deformations whenever necessary, we may suppose that $S \cap hS$ consists of at most a finite number of disjoint simple closed curves.

We shall call a component J of $S \cap hS$ *innermost on S* if the disc $D \subset S$ bounded by J has the property that $D \cap hS = J$. In this case we shall also say that D is an *innermost disc* on S. An innermost curve and an innermost disc on hS are defined analogously. Let J be a component of $S \cap hS$ such that J is innermost on hS, and let $D \subset hS$ be the disc bounded by J. Let D_1 and D_2 denote the two discs on S whose common boundary is J. Now, either $hD \subset D_1$ or $hD \subset D_2$, and we assume, without loss of generality, that $hD \subset D_1$. Since S does not bound a 3-ball in $R^1 \times S^2$, it is not difficult to see that either $S_1 = D \cup D_1$ or $S_2 = D \cup D_2$ cannot bound a 3-ball in $R^1 \times S^2$. We consider these two cases separately.

We first suppose that S_1 does not bound a 3-ball in $R^1 \times S^2$. If $hJ = J$, then

$$hS_1 = h(D \cup D_1) = D_1 \cup D = S_1.$$

By our construction of S_1, by a small deformation of S_1 we have $S_1 \cap S = \emptyset$ without loosing the property that $hS_1 = S_1$ and that S_1 does not bound a 3-ball in $R^1 \times S^2$. Now $S = 0 \times S^2$ splits $R^1 \times S^2$ into $(-1, 0] \times S^2$ and $[0, 1) \times S^2$ and we may assume, without loss of generality, that $S_1 \subset (0, 1) \times S^2$.

Let R^3 be obtained from $[0, 1) \times S^2$ by filling in $\partial\big([0, 1) \times S^2\big) = S$ with a 3-ball B^3. Then S_1 is a polyhedral 2-sphere in E^3 and, hence, S_1 bounds a 3-ball B_1^3 in E^3 such that $B^3 \subset \operatorname{int} B_1^3$. By the Combinatorial Annulus Theorem [5], $\operatorname{cl}(B_1^3 - B^3)$ is homeomorphic to $[0, 1] \times S^2$. Hence, S_1 is isotopic to $0 \times S^2$ in $R^1 \times S^2$.

If $hJ \neq J$, choose a simple closed curve $J' \subset D_1$ sufficiently close to J and such that the annulus $A \subset D_1$, bounded by J and J', has the property that $A \cap hS = J$. Let D' be a disc sufficiently close to D and such that

$$D' \cap hD' = D' \cap hS = \emptyset, \qquad D' \cap S = \partial D' = J'$$

and $S_1' = (D_1 - A) \cup D'$ does not bound a 3-ball in $R^1 \times S^2$. By construction, $S_1' \cap hS_1'$ is a strict subset of $S \cap hS$.

If S_2 does not bound a 3-ball in $R^1 \times S^2$ and $hJ \neq J$, then $S_2 \cap hS_2 = D_2 \cap hD_2$ is a strict subset of $S \cap hS$. If $hJ = J$, choose a simple closed curve $J' \subset D_2$ such that the annulus $A \subset D_2$, bounded by J and J_1', has the property that $A \cap hS = J$. Now choose a disc D' sufficiently close to D and such that

$$D' \cap hD' = D' \cap hS = \emptyset, \qquad D' \cap S = \partial D' = J'$$

and $S_2' = (D_2 - A) \cup D'$ does not bound a 3-ball in $R^1 \times S^2$. By construction, $S_2' \cap hS_2'$ is a strict subset of $S \cap hS$.

Thus, in either of the above two cases, there is always a 2-sphere S' which does not bound a 3-ball in $R^1 \times S^2$ and such that either $hS' = S'$ or $S' \cap hS'$ is a strict subset of $S \cap hS$. Furthermore, by using the previous argument for S_1, we see that S' is isotopic to S. Since the number of components of $S \cap hS$ is finite, by simply repeating the above procedure a finite number of times we can obtain a 2-sphere S'' such that S'' is isotopic to $0 \times S^2$ and either $hS'' = S''$ or $S'' \cap hS'' = \emptyset$.

For $S'' \cap hS'' = \emptyset$, let X, Y and Z denote the closures of the components of $R^1 \times S^2 - (S'' \cup hS'')$. Since S'' is isotopic to $0 \times S^2$, we may assume that X, Y and Z are homeomorphic to $(-1, -1/2] \times S^2$, $[-1/2, 1/2] \times S^2$ and $[1/2, 1) \times S^2$, respectively, whence, $hY = Y$. Let S^3 denote the 3-sphere obtained from Y by filling in $\partial Y = S'' \cup hS''$ with two 3-balls. Since h is a fixed point free involution of Y, h extends naturally to a fixed point free involution of S^3. It now follows from [1] that there is a 2-sphere $S''' \subset Y$ which separates S'' from hS'' and such that $hS''' = S'''$. This proves Lemma 1.

Henceforth, we let $S \subset R^1 \times S^2$ denote the 2-sphere of Lemma 1, and M_i $(i = 1, 2)$ the closed complementary domains of S in $R^1 \times S^2$. Since S is isotopic to $0 \times S^2$ in $R^1 \times S^2$, M_i is homeomorphic to $[0, 1) \times S^2$ for $i = 1, 2$.

LEMMA 2. *If $hM_i = M_i$, then there is an open annulus $A \subset R^1 \times S^2$ such that $hA = A$, $A \cap S$ is a simple closed curve, and A separates $R^1 \times S^2$ into two components, each homeomorphic to $R^1 \times B^2$.*

Proof. Let R^3 be obtained from M_1 by filling in $\partial M_1 = S$ with a 3-ball. Then h extends naturally to an involution of R^3 with one fixed point and we may extend h further to $R^3 \cup \{\infty\} = S^3$ by setting $h(\infty) = \infty$. By [2], there is a 2-sphere $S^2 \subset S^3$ orthogonal to S, invariant under h and containing the two fixed points. We set $A_1 = S^2 \cap M_1$ and let $A_2 \subset M_2$ be defined analogously. Furthermore, since $h\partial A_1 = \partial A_1 \subset S = \partial M_2$, we may choose A_2 such that $\partial A_1 = \partial A_2$. The open annulus $A = A_1 \cup A_2$ now satisfies the conclusion of Lemma 2.

THEOREM 1. *If h is a fixed point free involution of $R^1 \times S^2$, then $R^1 \times S^2/h$ is homeomorphic either to $R^1 \times P^2$ or to the open 3-dimensional Möbius band.*

Proof. If $hM_1 = M_2$, then $R^1 \times S^2/h$ may be viewed as being obtained from $[0, 1) \times S^2$ by identifying antipodal points on the 2-sphere $\partial([0, 1) \times S^2) = 0 \times S^2$. Thus, $R^1 \times S^2/h$ is homeomorphic to the open 3-dimensional Möbius band.

If $hM_1 = M_1$, then, by Lemma 2, there is an open annulus A which separates $R^1 \times S^2$ into two homeomorphic components N_1 and N_2. Since $hA = A$, either $hN_1 = N_1$ or $hN_1 = N_2$. However, since N_1 is homeomorphic to $R^1 \times B^2$ and h is fixed point free of period 2, $hN_1 = N_1$ is not possible. Therefore, $hN_1 = N_2$ and $R^1 \times S^2/h$ may be viewed as being obtained from $R^1 \times B^2$ by identifying antipodal points on $\partial(R^1 \times B^2) = R^1 \times \partial B^2$. Thus, $R^1 \times S^2/h$ is homeomorphic to $R^1 \times P^2$. This proves Theorem 1.

REFERENCES

[1] G. R. Livesay, *Fixed point free involutions on the 3-sphere,* Annals of Mathematics 72 (1960), p. 603-611.

[2] — *Involutions with two fixed points on the three-sphere,* ibidem 18 (1963), p. 582-593.

[3] G. X. Ritter, *Free Z_8 actions on S^3,* Transactions of the American Mathematical Society 181 (1973), p. 195-212.

[4] Y. Tao, *On fixed point free involutions of $S^1 \times S^2$,* Osaka Mathematical Journal 14 (1962), p. 145-152.

[5] E. C. Zeeman, *Seminar on combinatorial topology,* Mimeographed Notes, Institut des Hautes Etudes Scientifiques, Paris 1963.

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