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CONTINUOUS IMAGES OF ORDERED COMPACTA AND HEREDITARILY LOCALLY CONNECTED CONTINUA

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A *compactum* is a compact Hausdorff space and a *continuum* is a connected compactum. An *ordered compactum* (*ordered continuum*) is a compactum (continuum) K provided with a total ordering $\leqslant$ such that the topology of K is the order topology induced on K by $\leqslant$. If K is an ordered compactum with order $\leqslant$, and a and b are elements of K with $a \leqslant b$, then $[a, b]$, $[a, b)$, $(a, b]$, and (a, b) will have the usual meanings. Any Hausdorff space which is a continuous image of an ordered compactum (ordered continuum) will be called an *IOK* (*IOC*). A space X is *paraseparable* (*Suslinian*) if each collection of disjoint non-empty open sets (non-degenerate continua) in X is countable. In this paper* we will establish a characterization of paraseparable connected IOK's which contain no non-degenerate metric subcontinuum. We will also establish some properties of hereditarily locally connected continua that are obtainable as continuous images of ordered compacta.

If S is a net whose domain is the directed set D, then we will use the notation $\{S_a,\ a \in D\}$ for S. When dealing with sequences, N will always denote the set of natural numbers. Let X be a continuum and let K_0 be a non-degenerate subcontinuum of X. K_0 is called a *continuum of convergence* if there exists a net $K = \{K_a,\ a \in D\}$ of subcontinua of X converging to K_0 such that for all a and β in D either $K_a = K_\beta$ or $K_a \cap K_\beta = \emptyset$ and $K_a \cap K_0 = \emptyset$. If X is a space and $A \subseteq B \subseteq X$, then we will use the notation ∂B to denote the boundary of B in X, and the notation $\partial_B A$ to denote the boundary of A in the subspace B. For the proofs of the first three theorems see [8].

THEOREM 1. *If X is a compactum and $K = \{K_a,\ a \in D\}$ is a convergent net of closed subsets of X such that* $\lim K$ *is non-degenerate and such that for all a and β in D either $K_a = K_\beta$ or $K_a \cap K_\beta = \emptyset$ and $K_a \cap \lim K = \emptyset$,*

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then there exist a sequence $\{K_n, n \in N\}$ of elements of $\{K_a, a \in D\}$ and open sets U and V such that $\overline{U} \cap \overline{V} = \emptyset$, $\limsup K_n$ is non-degenerate, $K_i \cap K_j = \emptyset$ for all $i \neq j$, and

$$K_i \cap \limsup K_n = \emptyset, \quad K_i \cap U \neq \emptyset \quad \text{and} \quad K_i \cap V \neq \emptyset \quad \text{for all } i.$$

THEOREM 2. *If X is a compactum, A and B are disjoint closed subsets of X, and $\{K_n, n \in D\}$ is a net of closed subsets of X such that $K_n \cap A \neq \emptyset$ and $K_n \cap B \neq \emptyset$ for all n in D, then there exists a subnet $K' = \{K_{n_a}, a \in E\}$ of K such that $\liminf K' \neq \emptyset$ and $\limsup K'$ is non-degenerate.*

THEOREM 3. *The continuum X is hereditarily locally connected if and only if it contains no continuum of convergence.*

Mardešić ([2], Corollary 1, p. 425) has shown that if X is a connected IOK and X contains no non-degenerate metric subcontinuum, then $\dim X = \operatorname{Ind} X = \operatorname{ind} X = 1$. The next theorem further delineates the structure of such continua.

THEOREM 4. *If X is a connected IOK and X contains no non-degenerate metric subcontinuum, then X is hereditarily locally connected.*

Proof. Suppose that X is not hereditarily locally connected. By Theorem 3, X contains a continuum of convergence K_0. Let $K = \{K_a, a \in D\}$ be a net of subcontinua of X converging to K_0. By Theorems 1 and 2, there exists a sequence $\{K_n, n \in N\}$ of disjoint elements of $\{K_a, a \in D\}$ such that $K_i \cap \limsup K_n = \emptyset$ for every i and $\limsup K_n$ contains a non-degenerate continuum H.

Let

$$A = \limsup K_n \cup \bigcup_n K_n.$$

Since $\{K_n, n \in N\}$ is a sequence of disjoint closed sets, we have

$$A = \operatorname{Cl} \bigcup_n K_n$$

and, therefore, A is a closed set. Thus, A is an IOK.

Let

$$U = \bigcup_n K_n.$$

Then, clearly, U is an F_σ. Furthermore, $A - U = \limsup K_n$ since each K_i is disjoint from $\limsup K_n$. Thus, U is an open F_σ in A. Since A is an IOK, $\partial_A U$ is separable (see [2], Theorem 2, p. 426). However,

$$\partial_A U = \operatorname{Cl}_A U - U = \overline{U} - U = A - \bigcup_n K_n = \limsup K_n.$$

Hence $\limsup K_n$ is separable. Now, $\limsup K_n$ is a closed subset of X, so that $\limsup K_n$ is an IOK. Since H is a closed subset of $\limsup K_n$ and $\limsup K_n$ is a separable IOK, H is a separable IOK (see [5], Theorem 12,

p. 19). However, every separable connected IOK is second countable (see [9], Theorem 1, p. 417), and hence every separable connected IOK is metrizable. Therefore, H is a non-degenerate metric subcontinuum of X.

THEOREM 5. *If X is a paraseparable connected IOK and K is a subcontinuum of X with empty interior, then K is metrizable.*

Proof. Let $U = X - K$. Then U is an open set. Since X is paraseparable, U is an F_σ (see [5], Corollary 3, p. 13). Thus, ∂U is separable (see [2], Corollary 2, p. 427). Now, $\operatorname{Int} K = \emptyset$ and $K = X - U$, so that $\overline{U} = X$. Hence, $\partial U = K$ and, therefore, K is separable. However, K is a closed subset of X, and hence K is a separable connected IOK. It follows that K is metrizable (see [9], Theorem 1, p. 417).

Let X be a connected Hausdorff space and let $p \in X$. A space X is said to be *netlike at p* (*rational at p*) if for each q in $X - \{p\}$ there exists a finite set (countable closed set) which separates p and q. A space X is *netlike* (*rational*) if it is netlike (rational) at each of its points. Pearson ([7], Theorem 7, p. 48) and Ward, Jr. ([10], p. 183), have shown independently that every netlike continuum is an IOC. By combining this result with the theorems above, it is possible to establish the characterization mentioned at the beginning of this paper.

THEOREM 6. *Let X be a paraseparable continuum containing no non-degenerate metric subcontinuum. Then X is netlike if and only if X is an IOK.*

Proof. If X is netlike, then it follows immediately from the remarks above that X is an IOK. Suppose that X is an IOK but that X is not netlike. It follows from [12], p. 98, that X contains a non-degenerate continuum K such that if $x \in K$, then X is not netlike at x. Now, by Theorem 4, X is hereditarily locally connected, and hence K contains a non-degenerate ordered continuum H. Since X is paraseparable, it follows from Theorem 5 that $\operatorname{Int} H \neq \emptyset$. Let $p \in \operatorname{Int} H$ be such that p is not an end point of H. We claim that X is netlike at p. To see this, let $q \in X - \{p\}$. Clearly, there exist two points a and b of H such that $a < p < b$ and $(a, b) \subseteq \operatorname{Int} H - \{q\}$. It follows that $\{a, b\}$ separates p from q in X, so that X is netlike at p, which contradicts the fact that $p \in K$. Therefore, X is netlike.

We now proceed to establish some properties of connected hereditarily locally connected IOK's. Let X be a space, and let F be a family of subsets of X. F is called a *G-null family* if for each two open sets U and V in X with $\overline{U} \cap \overline{V} = \emptyset$ not more than a finite number of elements of F meet both U and V. If $\{F_\alpha, \alpha \in A\}$ is a family of disjoint subsets of X, then $\{F_\alpha, \alpha \in A\}$ is said to have *property* D if

$$F_\alpha \cap \operatorname{Cl} \bigcup \{F_\beta, \beta \neq \alpha\} = \emptyset$$

for all α in A. Finally, if X is a metric space and F is a family of subsets of X, then F is called a *null family* if for every $\varepsilon > 0$ not more than a finite number of elements of F have diameter greater than ε. If X is a compact metric space and F is a family of subsets of X, then it follows immediately that F is a G-null family if and only if it is a null family. The next theorem is proved in [8].

THEOREM 7. *If X is a continuum, then the following statements are equivalent*:

(1) *X is hereditarily locally connected.*

(2) *Every family of disjoint continua in X with property* D *is a G-null family.*

(3) *The components of every closed subset of X form a G-null family.*

THEOREM 8. *Let X be a connected hereditarily locally connected IOK. Then X is metrizable if and only if X is Suslinian and each arc in X is separable.*

Proof. We assume that X is non-degenerate. First, suppose that X is Suslinian and that each arc in X is separable. It follows from Zorn's Lemma that there exists a maximal collection M of disjoint arcs in X. Since X is Suslinian, it is clear that M is countable. Let $M = \{M_i, i \in N\}$.

Now, each element of M is an arc, and hence each element of M is separable. For each i, let D_i be a countable dense subset of M_i, and let

$$D = \bigcup_i D_i.$$

D is clearly countable. We claim that D is dense in X. For, suppose that there exists an open set U such that $U \cap D = \emptyset$. Since $U \cap M_i$ is M_i-open, we must have $U \cap M_i = \emptyset$ for each i. Let $x \in U$. Since X is locally connected, there exists a connected open set V such that $x \in V \subseteq U$. Let $y \in V$ be such that $x \neq y$ and let K be a continuum such that $x, y \in K \subseteq V$. Let H be an irreducible continuum in K from x to y. Then H is locally connected, and hence H is an arc. Since $H \subseteq U$, we have $H \cap M_i = \emptyset$ for each i. Thus, $M \cup \{H\}$ is a collection of disjoint arcs in X properly containing M, which contradicts the maximality of M. It follows that D is dense in X, so that X is separable. Therefore, X is metrizable (see [9], Theorem 1, p. 417).

Next, suppose that X is metrizable. Since X is compact, it is clear that each arc in X is separable. Let d be a metric on X which induces the topology of X, and for each $A \subseteq X$ let $\delta(A)$ denote the diameter of A with respect to d. Suppose that X is not Suslinian. Since X is hereditarily locally connected, there exists an uncountable collection G of disjoint arcs in X. If $H \in G$, then $\delta(H) > 0$, since arcs always are non-degenerate. Hence there exist an $\varepsilon > 0$ and an uncountable subset S of G such that, for each H in S, $\delta(H) > \varepsilon$. Let $H_1 \in S$. Now, since X is a metric space, X

has a countable base at H_1. It follows that there is an open set B_1 containing H_1 such that uncountably many elements of S fail to meet B_1. Let H_2 denote such an element of S. Since X also has a countable base at H_2, there is an open set B_2 containing H_2 such that uncountably many elements of S fail to meet $B_1 \cup B_2$. Let H_3 denote such an element. Continuing in this manner, we obtain a sequence $\{H_n, n \in N\}$ of disjoint elements of S and a sequence $\{B_n, n \in N\}$ of open sets such that, for each n,

$$H_n \subseteq B_n \quad \text{and} \quad H_{n+1} \cap \bigcup_{i=1}^{n} B_i = \emptyset.$$

Let $H_0 = \emptyset$ and, for each $n > 0$, let

$$W_n = B_n - \bigcup_{i=0}^{n-1} H_i.$$

Then $\{W_n, n \in N\}$ is a sequence of open sets such that, for each n, $H_n \subseteq W_n$ and $H_n \cap W_m = \emptyset$ for $n \neq m$. Thus, for each n,

$$\mathrm{Cl} \bigcup \{H_m, m \neq n\} \subseteq X - W_n \subseteq X - H_n,$$

so that $\{H_n, n \in N\}$ is a collection of disjoint continua in X with property D. Also, each $H_n \in S$ and, therefore, $\delta(H_n) > \varepsilon$, and hence $\{H_n, n \in N\}$ is not a null family. Since X is a compact metric space, $\{H_n, n \in N\}$ is not a G-null family and, therefore, X is not hereditarily locally connected. We conclude that X is Suslinian.

It is well known that if X is a non-degenerate hereditarily locally connected metric continuum, then $\dim X = \mathrm{Ind}\, X = \mathrm{ind}\, X = 1$. The next theorem gives an upper limit for $\mathrm{ind}\, X$ when X is a connected hereditarily locally connected IOK.

THEOREM 9. *If X is a connected hereditarily locally connected IOK, then* $\mathrm{ind}\, X \leqslant 3$.

Proof. Let $x \in X$, and let U be an open set containing x. Every IOK is locally peripherally metrizable (see [4], Theorem 5, p. 566) and, therefore, there exists an open set V such that $x \in V$, $\overline{V} \subseteq U$, and ∂V is metrizable. We claim that $\mathrm{ind}\, \partial V \leqslant 2$.

Let $\mathscr{F}$ denote the set of all non-degenerate components of ∂V. Now, ∂V is a closed subset of X, and so, by Theorem 7, the components of ∂V form a G-null family. Thus, $\mathscr{F}$ is a G-null family. Consider now $\mathscr{F}$ as a collection of continua in the subspace ∂V. Since ∂V is a closed subset of X, $\mathscr{F}$ is a G-null family in the subspace ∂V. However, ∂V is metrizable, and hence $\mathscr{F}$ is a null family in ∂V. It follows that $\mathscr{F}$ is countable. Now, each element of $\mathscr{F}$ is a hereditarily locally connected metric continuum. Thus, if $K \in \mathscr{F}$, then $\mathrm{Ind}\, K = 1$.

Let $F = \bigcup \mathscr{F}$ and $E = \partial V - F$. Now, F is a metric space. Since $\mathscr{F}$ is a countable covering of F with closed sets each with $\mathrm{Ind} \leqslant 1$, it follows from the Sum Theorem (see [6], Theorem II.1, p. 17) that $\mathrm{Ind}\, F \leqslant 1$.

Next, consider E. We claim that $\mathrm{Ind}E \leqslant 0$. First of all we will show that $\mathrm{ind}E \leqslant 0$. Let $y \in E$ and let W^* be an E-open set such that $y \in W^*$. There exists a ∂V-open set W such that $W^* = W \cap E$. Now, since $y \in E$, $\{y\}$ is a component of ∂V. Thus, there exists a ∂V-open set B such that $y \in B \subseteq W$ and $\partial_{\partial V} B = \emptyset$. Let $B^* = B \cap E$. Then B^* is an E-open set and $y \in B^* \subseteq W^*$. Furthermore, $\partial_E B^* = \partial_{\partial V} B = \emptyset$, and hence $\mathrm{ind}E \leqslant 0$. Now, ∂V is a compact metric space and, therefore, every subset of ∂V is separable. Thus, E is a separable metric space. However, on any separable metric space, dim, ind and Ind all agree (see [6], Theorem IV.1, p. 90) and, therefore, $\mathrm{Ind}E \leqslant 0$.

Thus, $\partial V = E \cup F$, $\mathrm{Ind}E \leqslant 0$, and $\mathrm{Ind}F \leqslant 1$. It follows from the Decomposition Theorem (see [6], Theorem II.4, p. 19) that $\mathrm{Ind}\partial V \leqslant 2$. Since $\mathrm{ind}\partial V \leqslant \mathrm{Ind}\partial V \leqslant 2$, we conclude that $\mathrm{ind}X \leqslant 3$.

THEOREM 10. *Every Suslinian continuum is paraseparable.*

Proof. Let X be a continuum, and suppose that X is not paraseparable. Let $\{U_a, a \in A\}$ be an uncountable collection of disjoint non-empty open subsets of X. For each a in A, let V_a be an open set such that $\overline{V}_a \subseteq U_a$, and let K_a be a component of V_a. Then $\overline{K}_a$ is a continuum. Furthermore, $\overline{K}_a$ meets ∂V_a for each a in A. Thus, $\{\overline{K}_a, a \in A\}$ is an uncountable collection of non-degenerate subcontinua of X. Since $\overline{K}_a \subseteq U_a$ for each a, $\{\overline{K}_a, a \in A\}$ is an uncountable collection of disjoint non-degenerate subcontinua of X. Therefore, X is not Suslinian.

Whyburn ([11], p. 381) has proved that every hereditarily locally connected metric continuum is rational. It is an open question whether the same theorem holds for non-metric hereditarily locally connected continua. Our final theorem gives an interesting decomposition into rational continua for a certain class of hereditarily locally connected continua.

THEOREM 11. *Every Suslinian connected hereditarily locally connected IOK is the union of a totally disconnected set and a countable collection of rational continua.*

Proof. Suppose that X is not rational. Let R be the set of all points of X at which X is rational. If $x \in X - R$, then there exists a non-degenerate continuum K_x containing x such that X is rational at no point of K_x (see [12], p. 98). Clearly, $X - R = \bigcup \{K_x, x \notin R\}$. Let $A = X - R$. Consider $\overline{A}$. The continuum X is Suslinian, and $\overline{A}$ is a compact subset of X, and hence $\overline{A}$ has only countably many non-degenerate components. Let $\{K_i, i \in N'\}$, where $N' \subseteq N$, be the set of non-degenerate components of $\overline{A}$. Each K_i is a Suslinian connected hereditarily locally connected IOK. Let $i \in N'$, and suppose that H is an ordered continuum in K_i such that $\mathrm{Int}H \neq \emptyset$. Since $H \subseteq \overline{A}$, there exists a point p in $\mathrm{Int}H \cap A$. We claim that X is rational at p. Let $q \in X - \{p\}$. There exist points a and b in H

such that $a < b$, $p \in (a, b)$, $q \notin (a, b)$, and (a, b) is open. It follows that $\{a, b\}$ separates p and q in X so that X is rational at p, which contradicts the fact that $p \in A$. Thus, $\operatorname{Int} H = \emptyset$ for every arc H in K_i. By Theorem 10, K_i is paraseparable and, therefore, by Theorem 5, each arc in K_i is metrizable. Thus, it follows from Theorem 8 that each K_i is a hereditarily locally connected metric continuum, and hence that each K_i is a rational continuum. Let

$$F = \bigcup_i K_i \quad \text{and} \quad E = \bar{A} - F.$$

Clearly, E is totally disconnected, and hence $\bar{A}$ is the union of a totally disconnected set and a countable collection of rational continua.

Let $V = X - \bar{A}$. Now, V is an open set and X is a paraseparable IOK. Thus, V is an F_σ (see [5], Corollary 3, p. 13), i.e.,

$$V = \bigcup_i F_i,$$

where each F_i is a closed set. Let $i \in N$. For each x in F_i there exists a connected open set U_x^i such that $x \in U_x^i$ and $\overline{U_x^i} \subseteq V$. Now, since $\{U_x^i, x \in F_i\}$ covers F_i and F_i is compact, there exists a finite subset $\{U_{x_1}^i, \ldots, U_{x_n}^i\}$ of $\{U_x^i, x \in F_i\}$ such that

$$F_i \subseteq \bigcup_{j=1}^n U_{x_j}^i \quad \text{and} \quad \bigcup_{j=1}^n \overline{U_{x_j}^i} \subseteq V.$$

Let $E_{ij} = \overline{U_{x_j}^i}$ for all i and j. Then each E_{ij} is a non-degenerate continuum and, clearly,

$$\bigcup_{i,j} E_{ij} = V.$$

Now, X is rational at each point of V since $V \subseteq R$. From $E_{ij} \subseteq V \subseteq R$ it follows that each E_{ij} is a rational continuum. Since

$$X = E \cup \bigcup_i K_i \cup \bigcup_{i,j} E_{ij},$$

the theorem follows.

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