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THE FOURIER-STIELTJES ALGEBRA OF A SEMISIMPLE GROUP

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Let G be a locally compact group, and $B(G)$ its Fourier-Stieltjes algebra, as described by Eymard [4]. Then $B(G)$ is the space of all matrix coefficients

$$g \mapsto \langle \pi(g)\xi, \eta\rangle \qquad (\xi, \eta \in \mathscr{H}_\pi)$$

of unitary representations π of G on Hilbert spaces $\mathscr{H}_\pi$, which is an algebra of functions under pointwise operations because the direct sum and tensor product of two unitary representations is again a unitary representation, and which is closed under complex conjugation (written$^-$) because the contragradient of a unitary representation is also unitary. Moreover, equipped with the norm

$$\|u\|_B = \min\{\|\xi\|\,\|\eta\|:\ u = \langle \pi\xi, \eta\rangle\},$$

$B(G)$ is a Banach algebra.

The Fourier algebra $A(G)$ of G is an interesting ideal of $B(G)$ with many equivalent characterizations. We define $A(G)$ to be the space of all matrix coefficients of the regular representation of G on $L^2(G)$, $L^2(G)$ being the usual Lebesgue space on G relative to left Haar measure, and then $A(G)$ can be described as the closure in $B(G)$ of $L^2(G) \cap B(G)$. An important property of the semisimple Banach algebra $A(G)$ is that its maximal ideal space can be identified naturally with G: we write δ_g for the multiplicative linear functional $u \mapsto u(g)$. We shall use results of Eymard [4] without further comment, and we shall also cite without reference some results on unitary representations to be found in [3].

If G is Abelian, $A(G)$ and $B(G)$ are the images under the Fourier transformation of the convolution algebras of integrable functions and bounded measures on the dual group of G. These spaces have intensively been studied, as a glance at the references in [6] or [7] will attest. A familiar result of Williamson [8] is that $B(G)$ is symmetric if and only if G is compact. (We remind the reader that a Banach algebra of functions is *symmetric* if $\delta(\bar{u}) = \delta(u)^{-}$ for all multiplicative linear functionals δ and all functions u.)

If G is not Abelian, things are certainly different. Liukkonen and Mislove [5] have shown that, for certain non-compact motion groups, $B(G)$ is symmetric. Here we prove a similar result for connected semisimple Lie groups with finite centre.

Let G be a semisimple Lie group, connected with finite centre. It is well known that G has a finite covering group G° of the form

$$H_0^\circ \times H_1^\circ \times \ldots \times H_n^\circ,$$

where H_0° is compact and H_j° is non-compact and simple. Let D be the finite central subgroup of G° such that $G = G^\circ/D$. Let H_j be the image of H_j° in G, i.e., $H_j^\circ D/D$. Let $\mathscr{S}$ be the set of subgroups S of G which are the images of subgroups S° in G° of the form

$$H_{j_1}^\circ \times H_{j_2}^\circ \times \ldots \times H_{j_m}^\circ \quad \text{with } \emptyset \subseteq \{j_1, j_2, \ldots, j_m\} \subseteq \{1, 2, \ldots, n\}.$$

THEOREM. *Let G be a connected semisimple Lie group with finite centre and let $\mathscr{S}$ be as described above. Then the maximal ideal space Δ of $B(G)$ is*

$$\Delta = \bigcup_{S \in \mathscr{S}} G/S,$$

G is dense in Δ and $B(G)$ is symmetric.

Proof. In the first part of the proof, we consider irreducible unitary representations π of G. We let S be the largest subgroup in $\mathscr{S}$ on which π is trivial, and then π can be viewed as a representation of G/S. We show that there exist a constant independent of π and an integer q such that the matrix coefficients $\langle \pi\xi, \eta\rangle$, as functions on G/S, belong to $L^{2q}(G/S)$, and

$$\|\langle \pi\xi, \eta\rangle\|_{2q} \leqslant C\|\xi\|\,\|\eta\|.$$

The second part of the proof treats $B(G)$. We show that $B(G)$ is the direct sum of subalgebras $B_S(G)$ ($S \in \mathscr{S}$) of $B(G)$-functions constant on cosets of S but no larger S' in $\mathscr{S}$ and that the maximal ideal space of $B_S(G)$ is G/S.

Finally, this information is used to prove the theorem. Without further ado, we take the first step.

Let π be an irreducible unitary representation of G. Then π "lifts" to an irreducible unitary representation of G°, which is the external tensor product of irreducible unitary representations π_0 and π_j of H_0° and H_j°. According to results of the author [2], either π_j is the trivial representation of H_j° or there exists a finite p_j such that all the matrix coefficients of π_j lie in $L^{p_j}(H_j^\circ)$. Let S° be the product of the H_j° for which π_j is trivial, and let p be the maximum of the p_j. Then π can be viewed as a representation of G°/S°.

Now $\mathscr{H}_\pi$ can be viewed as the Hilbertian tensor product of Hilbert spaces $\mathscr{H}_{\pi_j}$. Let ξ be a vector of the form $\zeta_0 \otimes \zeta_1 \otimes \ldots \otimes \zeta_n$. Then $\langle \pi\xi, \xi \rangle$, viewed on G°, can be written as a product

$$\langle \pi\xi, \xi \rangle(h_0, h_1, \ldots, h_n) = \prod_1^n \langle \pi_j(h_j)\zeta_j, \zeta_j \rangle,$$

and this function, now viewed as a function on G°/S°, lies in $L^p(G^\circ/S^\circ)$. Further, π, viewed as a representation of G°/S°, is irreducible. Thus it follows from [1] that all matrix coefficients of π lie in some $L^{2q}(G^\circ/S^\circ)$ and

$$\|\langle \pi\theta, \eta \rangle\|_{2q} \leqslant C\,\|\theta\|\,\|\eta\|, \tag{1}$$

where C does not depend on π. Finally, π is constant on cosets of D in G°, so the estimate above can be transferred to the quotient group $(G^\circ/S^\circ)/(DS^\circ/S^\circ)$, i.e., to G/S, completing the first step.

The second step begins with the observation (cf. [1]) that the set $\hat{G}_{Sq}$ of irreducible unitary representations π of G (more properly, equivalence classes thereof) with the property that π is trivial on S and that (1) holds on G/S is closed in G in the Fell topology. As a consequence, the unitary dual $\hat{G}$ of G is the disjoint union of the sets $\hat{G}_S$, each of which is the union of the increasing family of closed sets $\hat{G}_{Sq}$ as q ranges over the positive integers.

We continue by observing that any unitary representation π of G on a separable Hilbert space can be expressed as a direct integral. We can therefore decompose π:

$$\pi = \sum_{S\in\mathscr{S}} \pi_S,$$

where π_S is the part of π supported in $\hat{G}_S$. We can now decompose any u in $B(G)$ in the obvious manner:

$$u = \sum_{S\in\mathscr{S}} u_S.$$

Thus we have a decomposition of $B(G)$:

$$B(G) = \sum_{S\in\mathscr{S}}^{\oplus} B_S(G).$$

A function u in $B_S(G)$ has a direct integral decomposition

$$u = \int_{\hat{G}_S} d\mu(\pi) \sum_1^{j(\pi)} \langle \pi\xi_\pi^k, \eta_\pi^k \rangle,$$

where

$$\int_{\hat{G}_S} d\mu(\pi) \sum_1^{j(\pi)} \|\xi_\pi^k\| \|\eta_\pi^k\| < \infty .$$

Since $\hat{G}_S$ is the union of the increasing family of closed sets $\hat{G}_{Sq}$ and μ is a regular Borel measure, for any small positive ε we can find a q such that

$$\int_{\hat{G}_S/\hat{G}_{Sq}} d\mu(\pi) \sum_1^{j(\pi)} \|\xi_\pi^k\| \|\eta_\pi^k\| < \varepsilon .$$

Let u_q be the part of u supported by $\hat{G}_{Sq}$; then

$$\|u - u_q\|_B < \varepsilon .$$

Moreover, u_q lies in $L^{2q}(G/S)$ (see [1]), and a fortiori to $C_0(G/S)$. Since u can be approximated by these u_q in $B(G/S)$ and hence uniformly, u itself is in $C_0(G/S)$.

We define $C_S(G)$ to be the space of continuous functions on G which are constant on cosets of S in G and which, as functions on G/S, vanish at infinity, and then

$$B_S(G) = B(G) \cap C_S(G) .$$

It is now clear that $B_S(G)$ is a Banach algebra and that

$$B_S(G) \cdot B_{S'}(G) \subseteq B_{S \cap S'}(G) . \tag{2}$$

To complete the second step, we note that if δ is a non-zero multiplicative linear functional on $B_S(G)$, then $\delta(u)$ is non-zero for some u in $B_S(G)$, and so, by the continuity of δ, $\delta(u_q)$ is non-zero for some positive integer q. Now $\delta(u_q)^q = \delta(u_q^q)$ and u_q^q, as a function on G/S, is square-integrable, hence in $A(G/S)$. Thus δ is non-trivial on $A(G/S)$, where it must be a point evaluation δ_{gS}. Pick u in $A(G/S)$ with $u(gS) = 1$. For v in $B_S(G)$, viewed as a function on G/S, uv is in $A(G/S)$, so that

$$\delta(v) = \delta(uv)/\delta(u) = \delta_{gS}(uv)/\delta_{gS}(u) = v(gS) . \tag{3}$$

Consequently, the maximal ideal space of $B_S(G)$ is contained in G/S; on the other hand, point evaluations δ_{gS} are multiplicative linear functionals on $B_S(G)$, so that the maximal ideal space of $B_S(G)$ is exactly G/S. The second step is completed.

For further investigation, we suppose that δ is a multiplicative linear functional on $B(G)$ which does not annihilate $B_S(G)$. Then δ is a non-trivial multiplicative linear functional restricted to $B_S(G)$, where it must be a point evaluation δ_{gS}. Let $B_{S+}(G)$ be the space of $B(G)$-functions constant on cosets of S in G. Repetition of the argument in (3) shows that δ is also the point evaluation δ_{gS} on $B_{S+}(G)$.

Now we can decompose Δ. Let δ be a non-trivial multiplicative linear functional on $B(G)$. If δ is non-zero on $B_S(G)$ and on $B_{S'}(G)$, then (2) shows that δ is also non-zero on $B_{S\cap S'}(G)$. There is therefore a smallest S such that δ does not annihilate $B_S(G)$, and that δ annihilates $B_{S'}(G)$ unless $S \subseteq S'$, that is, δ is non-trivial on $B_{S+}(G)$, where it acts as a point evaluation $\delta_{gS'}$, and δ annihilates the rest of $B(G)$. Let Δ_S be the space of such functionals. The Δ_S is contained in G/S. On the other hand, given gS in G/S, the multiplicative linear functional δ_{gS} can be defined by

$$\delta_{gS}(u) = \begin{cases} u(g) & \text{for } u \in B_{S+}(G), \\ 0 & \text{for } u \in B_{S'}(G),\ S' \not\supseteq S. \end{cases}$$

We have, therefore, an identification of Δ_S with G/S and of Δ with $\bigcup_{S\in\mathscr{S}} G/S$.

Now we consider the density of G in Δ; of course, we identify G with $\Delta_{\{e\}}$. For a non-trivial S, we suppose that S° is $H^\circ_{j_1} \times H^\circ_{j_2} \times \ldots \times H^\circ_{j_m}$, and let (g_r) be a sequence in S° whose projections on each of the factors H°_j $(j = j_1, j_2, \ldots, j_m)$ go to infinity. Let (g_r) be the sequence of images in G. If u is in $B_{S+}(G)$, then

$$\delta_{gS}(u) = u(g) = \lim_{r\to\infty} u(gg_r) = \lim_{r\to\infty} \delta_{gg_r}(u).$$

However, if u is in $B_{S'}(G)$ and $S \not\subseteq S'$, then u is constant on cosets of $S\cap S'$ but vanishes at infinity in the directions H_j, where $H_j \subseteq S$ and $H_j \not\subseteq S'$. It then follows that

$$\lim_{r\to\infty} \delta_{gg_r}(u) = 0 = \delta_{gS}(u).$$

So δ_{gS} is the limit of the δ_{gg_r}, and G is dense in Δ.

Finally, we discuss possible pathology of $B(G)$. Since G is dense in Δ, $\delta(\bar{u}) = \delta(u)^-$ for any u in $B(G)$ and any δ in Δ. Thus $B(G)$ is symmetric. S. W. Drury pointed out to me that another consequence of the density of G in Δ is the absence of a Wiener-Pitt phenomenon: if u in $B(G)$ takes positive real values, bounded away from zero, then u is invertible in $B(G)$.

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