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REMARKS ON A GAME OF CHOQUET

BY

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The object of this paper is the strong game of G. Choquet [2], further studied by E. Porada [11], F. Topsøe [17], and the author [14, 15, 16]. The paper is divided into two sections. Section 1 deals with metrizable spaces and, according to the complete description of winning sets for both players, it contains various solutions of the determinacy problem depending on extra set-theoretic assumptions. In particular, the Borel determinacy problem P 1148, posed by E. Porada [11], is solved affirmatively. Section 2 deals with non-metrizable spaces and contains various results and open problems concerning the winning sets of the players.

For the topological background the reader is referred to the monograph of R. Engelking [4].

The strong game of G. Choquet [2, 3], denoted here by $\mathrm{Ch}(X, Y)$, is defined as follows. There are given a topological space X and a subset Y of X. Player I chooses a point x_1 in Y and its open nbhd (= neighbourhood) U_1 in X, and then Player II chooses an open nbhd V_1 of x_1 in X with $V_1 \subset U_1$. Now Player I chooses a point x_2 in $V_1 \cap Y$ and its open nbhd U_2 with $U_2 \subset V_1$, and then Player II chooses an open nbhd V_2 of x_2 with $V_2 \subset U_2$, and so on. Player II wins the play $((x_1, U_1), V_1, (x_2, U_2), V_2, \ldots)$ of $\mathrm{Ch}(X, Y)$ if $Y \cap \bigcap_{n=1}^{\infty} V_n \neq \emptyset$, and otherwise Player I wins. (Notice that in [2, 3, 17] the players I and II are called β and α respectively.)

E. Porada [11] considered a game, denoted here by $\mathrm{P}(X, Y)$, which he called the game of Choquet, but $\mathrm{P}(X, Y)$ is slightly different from $\mathrm{Ch}(X, Y)$. The game $\mathrm{P}(X, Y)$ is played in the same way as $\mathrm{Ch}(X, Y)$, but Player II wins if $\emptyset \neq \bigcap_{n=1}^{\infty} V_n \subset Y$, and otherwise Player I wins.

Clearly, the games $\mathrm{Ch}(X, X)$ and $\mathrm{P}(X, X)$ coincide. Let $\mathrm{Ch}(X)$ denote the game $\mathrm{Ch}(X, X)$. Moreover, it is easy to check that if X is a metrizable space, then the games $\mathrm{Ch}(X, Y)$ and $\mathrm{P}(X, Y)$ are equivalent. In general, however these games are not equivalent. For, let X be any space containing a point x such that $\{x\}$ is not a G_δ-set in X. Then Player II has a winning strategy (w.s. for short) in $\mathrm{Ch}(X, \{x\})$, but Player I has a w.s. in $P(X, \{x\})$.

1. The games in metrizable spaces. In this section X is assumed to be a metrizable space and therefore we do not need to distinguish between $\mathrm{Ch}(X, Y)$ and $\mathrm{P}(X, Y)$. Moreover, for metrizable space X the game $\mathrm{Ch}(X, Y)$ is equivalent to $\mathrm{Ch}(Y)$.

THEOREM 1.1 (G. Choquet [2], p. 146). *Player II has a w.s. in* $\mathrm{Ch}(X, Y)$ *iff* Y *is an absolute* G_δ*-set (i.e.,* Y *is completely metrizable).*

In a characterization theorem for Player I, proved by E. Porada [11], Theorem 5.3, there are, however, two minor errors. First, the set S considered on p. 349 is not compact; in fact, S is a nowhere locally compact G_δ-set in N_*^ω, and thus S is homeomorphic to the space P of irrational numbers. Second, the inclusion $h(S) \subset X$ on p. 351 (in Theorem 5.3) should be replaced by $h(S) \subset \tilde{X}$, where $\tilde{X}$ is the completion of X. When introducing the above corrections and taking account on the facts that $h(T)$ is homeomorphic to the space Q of rational numbers and $h(T)$ is a G_δ-set in X, we get the following

THEOREM 1.2. *Player I has a (stationary) w.s. in* $\mathrm{Ch}(X, Y)$ *iff* Y *contains a copy* Z *of the space* Q *so that* Z *is a* G_δ*-set in* Y.

After this we are now ready to make use of some auxiliary results to characterize better the winning sets of Player I. Recall that X is said to be a *strongly Baire space* (or an F_{II}*-space*) if each closed subset of X is a set of the second category in itself. W. Hurewicz proved in [8] that a metrizable space is a strongly Baire space iff it does not contain a closed subset homeomorphic to Q. Moreover, it is easy to check that if Z is a G_δ-set in a T_1-space Y so that Z is homeomorphic to Q, then $\bar{Z}$ is a set of the first category in itself. From these considerations we get

THEOREM 1.3. *The following conditions are equivalent*:

(a) *Y is not a strongly Baire space.*

(b) *Y contains a closed subset Z homeomorphic to Q.*

(c) *Y contains a G_δ-set Z homeomorphic to Q.*

(d) *Player I has a w.s. in* $\mathrm{Ch}(Y)$ *(and also in* $\mathrm{Ch}(X, Y)$*).*

(e) *Player I has a stationary w.s. in* $\mathrm{Ch}(Y)$ *(and also in* $\mathrm{Ch}(X, Y)$*).*

Note that in both Theorems, 1.1 and 1.3, the existence of a w.s. does not depend on the space X. Moreover, if a player has a w.s., he also has a stationary w.s.

Now we refer to another result of W. Hurewicz [8]: If Y is a CA-set in a Polish space X, then Y is a strongly Baire space iff Y is a G_δ-set in X. Hence, by Theorems 1.1 and 1.3, we get the solution of the Borel determinacy problem posed by E. Porada ([11], p. 353, P 1148), and even more:

COROLLARY 1.4. *If Y is a CA-set in a Polish space X, then the game* $\mathrm{Ch}(X, Y)$ *is determined.*

The answer for analytic non-Borel sets depends, however, on the extra axioms of set theory. In the sequel we shall deal with various extensions of ZFC or ZF.

Notice that if Y is a subset of a Polish space X so that $X-Y$ is totally imperfect, then Y is a strongly Baire space, i.e., Y does not contain a relatively closed subset Z homeomorphic to Q. For, if Z is a copy of Q contained in Y, then $\bar{Z}-Z \not\subset X-Y$, because $\bar{Z}-Z$ is an uncountable G_δ-set and $X-Y$ is totally imperfect. Thus $\bar{Z}\cap Y \neq Z$, i.e., Z is not relatively closed in Y.

It was announced by K. Gödel [6] and proved by P. S. Novikov [10] that the axiom of constructibility implies the existence in the closed unit interval J of an analytic set whose complement is totally imperfect. Hence we get

COROLLARY 1.5. *Assuming the axiom of constructibility there is an analytic set* $Y \subset J$ *such that the game* $\mathrm{Ch}(J, Y)$ *is not determined.*

On the other hand, it follows from the results of J. R. Steel [13] and L. A. Harrington [7] that the axiom of analytic determinacy (for the binary game) is equivalent both to the proposition that all analytic non-Borel sets in J are Borel isomorphic and also to the proposition that every analytic non-Borel set in J is universal, in the generalized sense, for the analytic sets of J. Clearly, each analytic non-Borel set in J that is universal, in the generalized sense, for the analytic sets in J, contains a relatively closed subset homeomorphic to Q (cf. [12]). Hence we get

COROLLARY 1.6. *Assuming the axiom of analytic determinacy (for the binary game), the game* $\mathrm{Ch}(J, Y)$ *is determined for each analytic set* $Y \subset J$.

Furthermore, V. G. Kanoveĭ announced in [9] that the statement "each absolutely projective strongly Baire separable metric space is an absolute G_δ" is consistent with ZFC (Theorem 3), and the statement "each strongly Baire separable metric space is an absolute G_δ" is consistent with ZF+DC (Theorem 4). Hence we get

COROLLARY 1.7. *The statement "the game* $\mathrm{Ch}(X, Y)$ *is determined for each absolutely projective subset* Y *of a separable metric space* X*" is consistent with* ZFC.

COROLLARY 1.8. *The statement "the game* $\mathrm{Ch}(X, Y)$ *is determined for each subset* Y *of a separable metric space* X*" is consistent with* ZF+DC.

Notice that the axiom of choice yields the existence of Bernstein sets in J. Moreover, each Bernstein set in J is a strongly Baire space which is not an absolute G_δ. Therefore DC in Corollary 1.8 cannot be strengthened to AC. Finally, the axiom of constructibility implies the existence of a Bernstein set $Y \subset J$ such that Y is an absolute PCA $\cap$ CPCA-set (see [6] and [10]). Thus the statement on $\mathrm{Ch}(X, Y)$ in Corollary 1.7 does not hold in the constructible universe.

2. The games in non-metrizable spaces. In this section each space is assumed to be completely regular. It turns out that without a suitable assumption on X and Y the games $\mathrm{Ch}(X, Y)$ and $\mathrm{P}(X, Y)$ point out various singular and accidental properties, because, in general, none of the players can force the result of a play to be a singleton.

For the next considerations we recall the notion of W_δ-set introduced by H. H. Wicke and J. M. Worrell, Jr. [18].

A subset Y of a space X is said to be a *W_δ-set* in X if there is an indexed family $\{W(t_1, \ldots, t_n): (t_1, \ldots, t_n) \in T^n, n \in N\}$ of open sets in X so that for each $(t_1, t_2, \ldots) \in T^N$

$$\overline{W(t_1, \ldots, t_n t_{n+1})} \subset W(t_1, \ldots, t_n),$$

$$Y \subset \bigcup \{W(t): t \in T\},$$

$$W(t_1, \ldots, t_n) \cap Y \subset \bigcup \{W(t_1, \ldots, t_n, t): t \in T\},$$

$$\bigcap \{W(t_1, \ldots, t_n): n \in N\} \subset Y,$$

where T is an index set and N is the set of positive integers.

A subset Y of a space X is said to be a *generalized G_δ-set* in X if for each $x \in Y$ there is a G_δ-set G in X with $x \in G \subset Y$.

Clearly, each G_δ-set is a W_δ-set, and each W_δ-set is a generalized G_δ-set.

The notion of W_δ-set is related to a game denoted by $\mathrm{WW}(X, Y)$. This game is played as $\mathrm{P}(X, Y)$ but Player II wins if $\bigcap_{n=1}^{\infty} V_n \subset Y$, and otherwise Player I wins. It is easy to verify the following

THEOREM 2.1. *Player II has a w.s. in* $\mathrm{WW}(X, Y)$ *iff Y is a W_δ-set in X.*

By a routine argument (cf. [14], Theorem 1) one can derive the following

THEOREM 2.2. *Player II has a w.s. in* $\mathrm{P}(X, Y)$ *iff Y is a W_δ-set in X and Player II has a w.s. in* $\mathrm{P}(Y, Y)$.

THEOREM 2.3. *Assume that Player II has a w.s. in* $\mathrm{P}(X, X)$ *and $\emptyset \neq Y \subset X$. Then Player II has a w.s. in* $\mathrm{P}(X, Y)$ *iff Y is a W_δ-set in X.*

By a general theorem of [5] any w.s. in 2.1 can be (equivalently) replaced by a stationary w.s. Hence one can derive the analogues of 2.2 and 2.3 for stationary strategies as follows.

THEOREM 2.4. *Player II has a stationary w.s. in* $\mathrm{P}(X, Y)$ *iff Y is a W_δ-set in X and Player II has a stationary w.s. in* $\mathrm{P}(Y, Y)$.

THEOREM 2.5. *Assume that Player II has a stationary w.s. in* $\mathrm{P}(X, X)$ *and $\emptyset \neq Y \subset X$. Then Player II has a stationary w.s. in* $\mathrm{P}(X, Y)$ *iff Y is a W_δ-set in X.*

From 2.2, in particular, we get

COROLLARY 2.6. *If X is countably compact and $\emptyset \neq Y \subset X$, then Player II has a (stationary) w.s. in* $P(X, Y)$ *iff Y is a W_δ-set in X.*

The case of compact X was studied in [15, 16] so we left the topics aside. From the above considerations it follows, however, that "modulo W_δ-sets" the study of strategic options of Player II in the games $Ch(X, Y)$ and $P(X, Y)$ is reduced to the one of $Ch(Y)$.

Recall that a subset Y of a space X is said to be *G_δ-dense* in X if $W \cap Y \neq \emptyset$ for every nonempty G_δ-set W in X.

The statements in Theorem 2.7 below are easy to verify.

THEOREM 2.7. 1° *If Y is G_δ-dense in X, then Y is dense in X.* 2° *If Y is G_δ-dense in X and X is metrizable, then $Y = X$.* 3° *If Y is G_δ-dense in X and Player II has a w.s. in* $Ch(X)$, *then Player II has a w.s. in* $Ch(Y)$. 4° *If Y is dense in $N^* = \beta N - N$, then Y is G_δ-dense in N^*.*

From 3° and 4° we get

Example 2.8. Let Y be a dense subset of $N^* = \beta N - N$. Then Player II has a (stationary) w.s. in $Ch(Y)$.

Consider the following completeness-type properties:

(1) Player II has a stationary w.s. in $Ch(X)$.

(2) Player II has a Markov w.s. in $Ch(X)$.

(3) Player II has a w.s. in $Ch(X)$.

(4) There is an open base $\mathscr{B}$ for X such that $\bigcap_{n=1}^{\infty} B_n \neq \emptyset$ whenever $B_n \in \mathscr{B}$ and $\overline{B_{n+1}} \subset B_n$ for each $n \in N$.

(5) There is a sequence $\mathscr{B}_1, \mathscr{B}_2, \ldots$ of open bases for X so that $\bigcap_{n=1}^{\infty} B_n \neq \emptyset$ whenever $k_1 < k_2 < \ldots$, $B_n \in \mathscr{B}_{k_n}$ and $\overline{B_{n+1}} \subset B_n$ for each $n \in N$.

(6) There is a sequence $\mathscr{B}_1, \mathscr{B}_2, \ldots$ of open bases for X so that $\bigcap_{n=1}^{\infty} B_n \neq \emptyset$ whenever $B_n \in \mathscr{B}_n$ and $\overline{B_{n+1}} \subset B_n$ for each $n \in N$.

THEOREM 2.9.

$$
\begin{tikzcd}
 & (1) \arrow[r, Rightarrow] & (2) \arrow[dr, Rightarrow] & \\
(4) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & (3) \\
 & (5) \arrow[r, Rightarrow] & (6) \arrow[uu, Rightarrow] \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

The above implications are easy to check. It is, however, an unsettled question which of these implications can be reversed. Among the completeness properties (1)–(6), the property (3) points out rather a peculiar behavior (cf. 3° above). Also, notice that none of (1)–(6) is a monotonic property in the sense of [1].

Now we shall consider strategic options of Player I. Again, by a general theorem of [5], we derive the following

THEOREM 2.10. *Player I has a w.s. in* $\mathrm{Ch}(X, Y)$ *iff Player I has a stationary w.s. in* $\mathrm{Ch}(X, Y)$ *of the form* $(x_1, U_1) = s(\emptyset)$ *and* $(x_{n+1}, U_{n+1}) = s(x_n, V_n)$ *for every* $n \in N$. *(The same also holds for* $\mathrm{Ch}(X)$.*)*

The next theorem provides a characterization of winning sets for Player I in $\mathrm{Ch}(X, Y)$.

THEOREM 2.11. *Player I has a w.s. in* $\mathrm{Ch}(X, Y)$ *iff there is a collection* $\{\mathscr{B}(U)$: *U is open in* $X\}$ *of families of open sets in X so that*

(a) $\mathscr{B}(U)$ *is a nbhd base of a point in* $U \cap Y$,

(b) $B \in \mathscr{B}(U)$ *implies* $\bar{B} \subset U$,

(c) $Y \cap \bigcap_{n=1}^{\infty} B_n = \emptyset$ *whenever* $B_1 \in \mathscr{B}(X)$, $B_2 \in \mathscr{B}(B_1)$, $B_3 \in \mathscr{B}(B_2), \ldots$

Similarly, for the game $\mathrm{P}(X, Y)$ we have

THEOREM 2.12. *Player I has a w.s. in* $\mathrm{P}(X, Y)$ *iff there is a collection* $\{\mathscr{B}(B_1, \ldots, B_n)$: $B_1 \in \mathscr{B}(\emptyset)$, $B_2 \in \mathscr{B}(B_1), \ldots, B_n \in \mathscr{B}(B_1, \ldots, B_{n-1})$, $n \in N\}$ *of families of open sets in X so that*

(a) $\mathscr{B}(\emptyset)$ *is a nbhd base of a point in Y and* $\mathscr{B}(B_1, \ldots, B_n)$ *is a nbhd base of a point in* $B_n \cap Y$,

(b) $B \in \mathscr{B}(B_1, \ldots, B_n)$ *implies* $\bar{B} \subset B_n$,

(c) *either* $\bigcap_{n=1}^{\infty} B_n = \emptyset$ *or* $\bigcap_{n=1}^{\infty} B_n \not\subset Y$ *whenever* $B_1 \in \mathscr{B}(\emptyset)$, $B_2 \in \mathscr{B}(B_1)$, $B_3 \in \mathscr{B}(B_1, B_2), \ldots$

Both characterizations, 2.11 and 2.12, are related to dual games $\mathrm{Ch}_1(X, Y)$ and $\mathrm{P}_1(X, Y)$ to the games $\mathrm{Ch}(X, Y)$ and $\mathrm{P}(X, Y)$ respectively. In $\mathrm{Ch}_1(X, Y)$ Player I chooses a nbhd base $\mathscr{B}_1$ of a point in Y, and then Player II chooses a $B_1 \in \mathscr{B}_1$. Now Player I chooses a nbhd base $\mathscr{B}_2$ of a point in $B_1 \cap Y$ with $\bar{B} \subset B_1$ for each $B \in \mathscr{B}_2$, and then Player II chooses a $B_2 \in \mathscr{B}_2$, and so on. Player II wins the play $(\mathscr{B}_1, B_1, \mathscr{B}_2, B_2, \ldots)$ if $Y \cap \bigcap_{n=1}^{\infty} B_n \neq \emptyset$, and otherwise Player I wins.

Hence, Theorem 2.11 can be rephrased as follows: Player I has a (stationary) w.s. in $\mathrm{Ch}(X, Y)$ iff Player I has a stationary w.s. in $\mathrm{Ch}_1(X, Y)$.

Similarly, Player II wins the play $(\mathscr{B}_1, B_1, \mathscr{B}_2, B_2, \ldots)$ of $\mathrm{P}_1(X, Y)$ if $\emptyset \neq \bigcap_{n=1}^{\infty} B_n \subset Y$, and otherwise Player I wins.

Hence, Theorem 2.12 is equivalent to the following statement: Player I has a w.s. in $\mathrm{P}(X, Y)$ iff Player I has a w.s. in $\mathrm{P}_1(X, Y)$. Note that the last statement remains true if both w.s. are replaced by stationary w.s.

THEOREM 2.13. *If Y is not a generalized* G_δ*-set in X, then Player I has a stationary w.s. in* $\mathrm{P}(X, Y)$.

For, let x be a point of Y so that there is no G_δ-set G in X with

$x \in G \subset Y$. Putting $s(\emptyset) = (x, X)$ and $s(V) = (x, V)$ for any open nbhd V of x, we get a stationary w.s. for Player I.

To avoid this trivial case in the study of $\mathrm{P}(X, Y)$ we may assume that Y is always a generalized G_δ-set in X. This assumption, however, is not very useful toward a characterization of winning sets for Player I. The following is a partial result proved by a standard argument.

THEOREM 2.14. *If $\emptyset \neq Z \subset Y \subset X$, where Z is of the first category in itself and is a W_δ-set in Y, then Player I has a stationary w.s. in* $\mathrm{P}(X, Y)$.

From 2.14 we get immediately

COROLLARY 2.15. *If X contains a nonvoid W_δ-set which is of the first category in itself, then Player I has a stationary w.s. in* $\mathrm{Ch}(X)$.

However, it is an unsettled question whether the converse to 2.15 (resp. 2.14) holds. **(P 1301)**

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