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ON THE BLOCH-NEVANLINNA CONJECTURE

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A function $f(z)$ analytic in $|z| < 1$ is said to be of *bounded characteristic* if

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta \leqslant M < \infty, \quad r < 1.$$

It is well known that such a function has a (finite) radial limit almost everywhere.

The "Bloch-Nevanlinna conjecture" asserts that if $f(z)$ is of bounded characteristic, so is $f'(z)$. Many counterexamples have been given. The purpose of this note is to offer two simple constructions which disprove the conjecture in a decisive way. The arguments are based on the following known lemma concerning the Rademacher functions $\varphi_n(t)$:

LEMMA. *If $\sum_{n=1}^{\infty} c_n z^n$ is analytic in $|z| < 1$ and $\sum_{n=1}^{\infty} |c_n|^2 = \infty$, then for almost every $t \in [0, 1]$, the function $\sum_{n=1}^{\infty} \varphi_n(t) c_n z^n$ has a radial limit almost nowhere (i.e., on no set of positive measure).*

Since the existence of a radial limit is equivalent to the Abel summability of the trigonometric series $\Sigma \varphi_n(t) c_n e^{in\theta}$, this lemma is a special case of a much more general result ([2], p. 214) asserting the non-summability of such series by any regular method of summability.

THEOREM 1. *There exists a function $f(z)$ analytic in $|z| < 1$ and continuous in $|z| \leqslant 1$, such that no fractional derivative $f^{(a)}(z)$ of positive order has a radial limit on any set of positive measure.*

If $g(z) = \sum_{n=1}^{\infty} b_n z^n$, its fractional derivative of order a will be defined as

$$g^{(a)}(z) = \sum_{n=1}^{\infty} \lambda_n b_n z^n,$$

where

$$\lambda_n = \frac{n!}{\Gamma(n+1-a)} \sim n^a.$$

This differs from the usual definition by an inessential factor z^a. To prove the theorem, let

$$f_t(z) = \sum_{n=1}^{\infty} \varphi_n(t)\, a_n z^n,$$

where $\varphi_n(t)$ is the n^{th} Rademacher function and

$$a_n = \begin{cases} 1/k^2, & n = 2^k \ (k = 1, 2, \ldots), \\ 0 & \text{otherwise}. \end{cases}$$

For every $t \in [0,1], f_t(z)$ is continuous in the closed disk, since $\Sigma |a_n| < \infty$. Let $a_j = 1/j$, $j = 1, 2, \ldots$ Then since $\Sigma n^{2a} |a_n|^2 = \infty$ for every $a > 0$, it follows from the lemma that for each $t \in [0,1]$ outside a set E_j of measure zero, the fractional derivative $f_t^{(a_j)}(z)$ has a radial limit almost nowhere.

We claim now that for each fixed $t \notin E = \bigcup_{j=1}^{\infty} E_j$, no fractional derivative $f_t^{(a)}(z)$ of positive order a can have a radial limit on any set of positive measure. For if $a_j < a, f_t^{(a_j)}(z)$ must have a radial limit wherever $f_t^{(a)}(z)$ does. To see this, we write

$$h(r) = f_t^{(a)}(re^{i\theta}) = \sum_{n=1}^{\infty} c_n r^n,$$

where $c_n = \varphi_n(t) \lambda_n a_n e^{in\theta}$. Under the assumption that $\lim_{r\to 1} h(r)$ exists we are to show that

$$H(r) = \sum_{n=1}^{\infty} \frac{\Gamma(n+1-a)}{\Gamma(n+1-a_j)} c_n r^n$$

also tends to a finite limit as $r \to 1$. But $H(r)$ may be written in the form

$$H(r) = \frac{1}{\Gamma(\beta_j)} \int_0^1 t^{-a} (1-t)^{\beta_j - 1} h(rt)\, dt,$$

where $\beta_j = a - a_j > 0$ and we assume without loss of generality that $a < 1$. Hence $H(r)$ has a limit if $h(r)$ does, by the Lebesgue bounded convergence theorem. This completes the proof.

Since the exceptional set E has measure zero, almost every choice of t gives a function $f = f_t$ with the required property.

A similar construction shows that the Bloch-Nevanlinna conjecture may fail even if $f(z)$ is very smooth on the boundary. We refer to the Zygmund class Λ^* which consists of all continuous functions $F(\theta)$ periodic with period 2π, such that

$$|F(\theta+h)-2F(\theta)+F(\theta-h)| \leqslant Ah$$

for some constant A and all $h > 0$. Thus Λ^* contains the Lipschitz class Λ_1, and it can be shown that $\Lambda^* \subset \Lambda_a$ for every $a < 1$. (See [2], p. 42 ff.)

THEOREM 2. *There exists a function $f(z)$ analytic in $|z| < 1$ and continuous in $|z| \leqslant 1$, such that $f(e^{i\theta}) \in \Lambda^*$, yet $f'(z)$ has a radial limit almost nowhere.*

We remark that the slightly stronger hypothesis $f(e^{i\theta}) \in \Lambda_1$ would imply $f'(z)$ is bounded, thus that it has a radial limit almost everywhere. To prove Theorem 2, let

$$a_n = \begin{cases} 1/n, & n = 2^k \ (k = 1, 2, \ldots), \\ 0 & \text{otherwise}, \end{cases}$$

and define $f_t(z) = \sum_{n=1}^{\infty} \varphi_n(t) a_n z^n$. Then

$$\sum_{n=1}^{N} n^2 |a_n| = O(N),$$

so $f_t(e^{i\theta}) \in \Lambda^*$ for every t. (See e.g. [1], p. 252.) But $\Sigma n^2 |a_n|^2 = \infty$, so the lemma asserts that for almost every t, $f_t'(z)$ has a radial limit almost nowhere.

REFERENCES

[1] P. L. Duren, H. S. Shapiro, and A. L. Shields, *Singular measures and domains not of Smirnov type*, Duke Mathematical Journal 33 (1966), p. 247-254

[2] A. Zygmund, *Trigonometric Series*, Vol. I, Cambridge 1959.

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