COLLOQUIUM MATHEMATICUM

VOL. LIII 1987 FASC. 1

ON EXTENSIONS OF MEASURES

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Some years ago C. Ryll-Nardzewski posed the following problem: Suppose $(X, \mathfrak{C})$ is a Borel space and μ is a measure on $\mathfrak{C}$. Suppose $\mathfrak{D}$ is a family of subsets of X such that for every countable subfamily $\mathfrak{D}_0$, μ has an extension to $\sigma(\mathfrak{C}, \mathfrak{D}_0)$. Then does μ admit an extension to $\sigma(\mathfrak{C}, \mathfrak{D})$?

It was proved in [2] that if $L_{\mathfrak{B}}$, the lattice of all substructures of $\mathfrak{B}$, is complemented then every countably generated (c.g. for short) substructure of $\mathfrak{B}$ is given by a countable partition. P13 of [2] raises the problem as to whether this property of $\mathfrak{B}$ implies that $L_{\mathfrak{B}}$ is complemented.

Recently Aniszczyk [1] showed that for any Borel space $(X, \mathfrak{B})$, $(*) \Leftrightarrow$ (a) and (b) where:

$(*)$ Every 0–1 measure defined on any substructure of $\mathfrak{B}$ can be extended as a 0–1 measure on $\mathfrak{B}$.

(a) Every c.g. substructure of $\mathfrak{B}$ is given by a countable partition.

(b) Every measure defined on any substructure of $\mathfrak{B}$ can be extended to a measure on $\mathfrak{B}$.

He raised the questions as to whether (a) $\Rightarrow$ $(*)$ and (b) $\Rightarrow$ $(*)$.

In this paper the above questions are settled as follows: The problem of Ryll-Nardzewski as well as P13 of [2] have negative solutions. (a) does not imply $(*)$. If $\mathrm{MA} + \aleph_1 < 2^{\aleph_0}$ holds then (b) implies $(*)$. If CH holds then (b) does not imply $(*)$.

We work in ZFC and follow the notation of [2].

LEMMA 1. *Let X be an uncountable set. Suppose for each ordinal $\alpha < \omega_1$, $\mathfrak{P}_\alpha$ is a countable partition of X such that for $\alpha < \beta$, $\mathfrak{P}_\beta$ is a refinement of $\mathfrak{P}_\alpha$. Let $\mathfrak{C}$ be the countable-cocountable structure on X and μ the 0–1 measure on $\mathfrak{C}$ giving measure 0 to countable sets. Let $\mathfrak{B} = \sigma(\mathfrak{C}, \bigcup_\alpha \mathfrak{P}_\alpha)$.*

Then

(i) *every c.g. substructure of $\mathfrak{B}$ is given by a countable partition;*

(ii) *μ on $\mathfrak{C}$ can be extended as a 0–1 measure on $\mathfrak{B}$ iff for each α, there is an uncountable set A_α in $\mathfrak{P}_\alpha$ such that $A_\beta \subset A_\alpha$ whenever $\alpha < \beta$.*

Proof. (i) If $\mathfrak{B}_0 \subset \mathfrak{B}$ is c.g. then there is a countable subfamily $\mathfrak{C}_0 \subset \mathfrak{C}$ and an α_0 such that $\mathfrak{B}_0 \subset \sigma(\mathfrak{C}_0, \mathfrak{P}_{\alpha_0})$ and the latter structure is given by a countable partition.

(ii) If μ can be extended as a 0–1 measure $\tilde{\mu}$ on $\mathfrak{B}$ then for each α there is exactly one set A_α in $\mathfrak{P}_\alpha$ such that $\tilde{\mu}(A_\alpha) = 1$. These sets satisfy our requirements.

To prove the converse, suppose A_α, $\alpha < \omega_1$, exist as stated. In case there is an α_0 such that $A_\beta = A_{\alpha_0}$ for all $\beta \geq \alpha_0$ then the structure $\mathfrak{B} \cap A_{\alpha_0}$ is the countable-cocountable structure on A_{α_0}. Define $\tilde{\mu}$ on $\mathfrak{B}$ by $\tilde{\mu}(B) = 0$ if $B \cap A_{\alpha_0}$ is countable and $\tilde{\mu}(B) = 1$ if $B \cap A_{\alpha_0}$ is cocountable in A_{α_0}. Then $\tilde{\mu}$ extends μ.

In case for every α there is a $\beta > \alpha$ such that $A_\beta \neq A_\alpha$, we proceed as follows. First observe that for every $B \in \mathfrak{B}$ there is an α such that $A_\alpha - \bigcap_\beta A_\beta \subset B$ or $A_\alpha - \bigcap_\beta A_\beta \subset B^c$. Also observe that exactly one of the above should hold, for, otherwise there are α_1 and α_2 such that

$$A_{\alpha_1} - \bigcap_\beta A_\beta \subset B \quad \text{and} \quad A_{\alpha_2} - \bigcap_\beta A_\beta \subset B^c$$

so that if $\alpha = \max(\alpha_1, \alpha_2)$ then $A_\alpha = \bigcap_\beta A_\beta$ which cannot occur in the case under consideration. Now define a 0–1 measure $\tilde{\mu}$ on $\mathfrak{B}$ by $\tilde{\mu}(B) = 1$ if for some α, $A_\alpha - \bigcap_\beta A_\beta \subset B$ and $\tilde{\mu}(B) = 0$ if for an α, $A_\alpha - \bigcap_\beta A_\beta \subset B^c$. To see that $\tilde{\mu}$ is countably additive, let $B_1, B_2, \ldots$ be sets in $\mathfrak{B}$ such that $\tilde{\mu}(B_i) = 0$ for every i. Then for each i there is an α_i such that $A_{\alpha_i} - \bigcap_\beta A_\beta \subset B_i^c$. Taking $\gamma > \alpha_i$ for all i observe that $A_\gamma - \bigcap_\beta A_\beta \subset (\bigcup_i B_i)^c$ proving that $\tilde{\mu}(\bigcup_i B_i) = 0$.

Since $\bigcap_\alpha (A_\alpha - \bigcap_\beta A_\beta) = \emptyset$, for any x there is an α_0 such that $x \notin A_{\alpha_0} - \bigcap_\beta A_\beta$ proving that $\tilde{\mu}\{x\} = 0$. This shows that $\tilde{\mu}$ is an extension of μ.

THEOREM 1. *Let X be a set of power $\aleph_1$. Let $\mathfrak{C}$ be the countable-cocountable structure on X and μ the 0–1 measure on $\mathfrak{C}$ giving measure 0 to singletons. Then there is a structure $\mathfrak{B}$ on X such that* (i) $\mathfrak{C} \subset \mathfrak{B}$, (ii) *every c.g. substructure of $\mathfrak{B}$ is given by a countable partition and* (iii) *μ cannot be extended as a 0–1 measure on $\mathfrak{B}$.*

Proof. Let $\leq$ be a partial order on X such that $(X, \leq)$ is an Aronszajn tree ([3], § 1), that is, (a) for every x in X, $\{y\colon y \leq x\}$ is well ordered with order type, say $\varrho(x)$, (b) for every x, $\varrho(x) < \omega_1$, (c) for every $\alpha < \omega_1$ the set $\{x\colon \varrho(x) = \alpha\}$ is countable and (d) there is no linearly ordered subset of power $\aleph_1$.

For each $\alpha < \omega_1$, let $\mathfrak{P}_\alpha$ be the partition of X consisting of the singleton sets $\{x\}$ for each x with $\varrho(x) < \alpha$ and the sets $\{y: x \leqslant y\}$ for each x with $\varrho(x) = \alpha$. This is a countable partition and for $\alpha < \beta$, $\mathfrak{P}_\beta$ is a refinement of $\mathfrak{P}_\alpha$. Let $\mathfrak{B} = \sigma(\bigcup_\alpha \mathfrak{P}_\alpha)$. (i) clearly holds. Lemma 1 proves (ii). To prove (iii) suppose for each α, A_α is an uncountable set in $\mathfrak{P}_\alpha$ such that $A_\beta \subset A_\alpha$ whenever $\alpha < \beta$. Then by the definition of $\mathfrak{P}_\alpha$ it must be the case $A_\alpha = \{y: x_\alpha \leqslant y\}$ for some x_α with $\varrho(x_\alpha) = \alpha$. Then $\{x_\alpha, \alpha < \omega_1\}$ is a linearly ordered set of power $\aleph_1$ contradicting (d). So, by (ii) of Lemma 1, μ cannot be extended as a 0–1 measure on $\mathfrak{B}$.

COROLLARY 1. *The problem of Ryll-Nardzewski has a negative solution.*

Proof. Let $\mathfrak{C}$, $\mathfrak{B}$, μ be as in Theorem 1. If $\mathfrak{D}_0$ is a countable subfamily of $\mathfrak{B}$ then, $\sigma(\mathfrak{C}, \mathfrak{D}_0) = \sigma(\mathfrak{C}, \sigma(\mathfrak{D}_0))$ and $\sigma(\mathfrak{D}_0)$ is given by a countable partition. As a consequence μ can be extended to $\sigma(\mathfrak{C}, \mathfrak{D}_0)$. Suppose now that μ has an extension $\tilde{\mu}$ to $\mathfrak{B}$. Then in view of the fact that every c.g. substructure of $\mathfrak{B}$ is given by a countable partition we conclude that $\tilde{\mu}$ is of the form $\sum_i a_i \tilde{\mu}_i$ where each $\tilde{\mu}_i$ is a 0–1 measure, $a_i > 0$ for all i and $\sum_i a_i = 1$. It is easy to see that $\tilde{\mu}_1$ is a 0–1 measure which extends μ. But such an extension does not exist by Theorem 1. As a consequence μ has no extension to $\mathfrak{B} = \sigma(\mathfrak{C}, \mathfrak{B})$.

COROLLARY 2. *Let* $(X, \mathfrak{B})$ *be as in Theorem* 1. *Then* $L_{\mathfrak{B}}$ *is not complemented.*

Since every c.g. substructure of $\mathfrak{B}$ is given by a countable partition, it follows that P13 of [2] has a negative solution.

Proof. Let $\mathfrak{C}$ be as in Theorem 1. Suppose there is a structure $\mathfrak{D}$ such that $\mathfrak{C} \vee \mathfrak{D} = \mathfrak{B}$ and $\mathfrak{C} \wedge \mathfrak{D} = \{\emptyset, X\}$. Fix a point x_0 in X. Define $\tilde{\mu}$ on $\mathfrak{B}$ by $\tilde{\mu}(B) = 1$ if there is a D in $\mathfrak{D}$ such that $B \Delta D$ is countable and $x_0 \in D$. $\tilde{\mu}(B) = 0$ if there is a D in $\mathfrak{D}$ such that $B \Delta D$ is countable and $x_0 \notin D$. Then $\tilde{\mu}$ is a 0–1 measure on $\mathfrak{B}$ extending μ of Theorem 1 (details are as in the proof of Proposition 39 of [2]). By Theorem 1 such an extension does not exist. So $\mathfrak{C}$ has no complement in $\mathfrak{B}$.

Remark. Aniszczyk informs us that he also has negative solution to P13 of [2].

COROLLARY 3. (a) *does not imply* (*).

Proof. The space $(X, \mathfrak{B})$ of Theorem 1 satisfies (a) but not (*).

Remarks. Let $(X, \mathfrak{B})$ be as in Theorem 1. Then every 0–1 measure on $\mathfrak{B}$ is concentrated at a point, as otherwise, it would be an extension of the μ of Theorem 1. Since every c.g. substructure of $\mathfrak{B}$ is given by a countable partition there is no nonatomic measure on $\mathfrak{B}$. As a consequence every measure on $\mathfrak{B}$ is concentrated on a countable subset. It follows that power set of $\aleph_1$ cannot support any continuous measure. This is a celebrated result of Ulam.

Now we shall investigate the validity of (b) $\Rightarrow$ (*).

THEOREM 2. *Assume* CH. *There is a Borel space* $(X, \mathfrak{B})$ *such that* (i) *every measure on any substructure of* $\mathfrak{B}$ *can be extended to* $\mathfrak{B}$ *and* (ii) *there is a* 0–1 *measure on a substructure which cannot be extended as a* 0–1 *measure on* $\mathfrak{B}$. *In other words* (b) *does not imply* (*).

Proof. Let $X \subset [0, 1]$ be an uncountable set such that every uncountable subset of X has positive Lebesgue outer measure. Assuming CH such a set can be constructed ([5], C_{26}). Let $\mathfrak{B}$ be the Borel structure of $[0, 1]$ relativized to X and λ be the normalized Lebesgue outer measure on $\mathfrak{B}$ and μ be any measure on $\mathfrak{C}$. We show that μ can be extended to $\mathfrak{B}$. Let $\mu = \mu_1 + \mu_2$ where $\mu_1 \ll \lambda$ on $\mathfrak{C}$ and $\mu_2 \perp \lambda$ on $\mathfrak{C}$. Note that for A in $\mathfrak{B}$ if $\lambda(A) = 0$ then A is countable. As a consequence μ_2 is concentrated on a countable set, and hence can be extended to $\mathfrak{B}$. If $f = \frac{d\mu_1}{d\lambda}$ on $\mathfrak{C}$ then the formula $\tilde{\mu}_1(B) = \int_B f\,d\lambda$ for $B \in \mathfrak{B}$ gives an extension of μ_1 to $\mathfrak{B}$. Thus (b) holds for $(X, \mathfrak{B})$.

If $\mathfrak{C} \subset \mathfrak{B}$ is the countable-cocountable structure and μ is the 0–1 measure on $\mathfrak{C}$ giving measure 0 to countable sets then μ cannot be extended to $\mathfrak{B}$ as a 0–1 measure because $\mathfrak{B}$ is separable. Thus (*) fails for $(X, \mathfrak{B})$.

THEOREM 3. *Assume* $\mathrm{MA} + \aleph_1 < 2^{\aleph_0}$. *Then for any Borel space* $(X, \mathfrak{B})$, (b) $\Rightarrow$ (*).

Proof. Let $(X, \mathfrak{B})$ be a Borel space satisfying (b). Let $\mathfrak{C}$ be a substructure of $\mathfrak{B}$ and μ be a 0–1 measure on $\mathfrak{C}$. We show that μ can be extended as a 0–1 measure on $\mathfrak{B}$. In view of (b) let $\tilde{\mu}$ be any extension. If $\tilde{\mu}$ has an atom A then the normalization of the restriction of $\tilde{\mu}$ to A serves as an extension of μ. If $\tilde{\mu}$ is nonatomic then there is a c.g. substructure $\mathfrak{B}_0 \subset \mathfrak{B}$ with uncountably many atoms. Now $(X, \mathfrak{B}_0)$ is a c.g. Borel space with uncountably many atoms and it has property (b). This is not possible in view of the following lemma.

LEMMA 2. *Assume* $\mathrm{MA} + \aleph_1 < 2^{\aleph_0}$. *Every uncountable separable Borel space does not satisfy* (b).

Proof. Let $(X, \mathfrak{B})$ be any uncountable separable Borel space. Fix $A \subset X$ of power $\aleph_1$. Define

$$\mathfrak{C} = \{B \in \mathfrak{B}\colon B \cap A \text{ is either countable or cocountable in } A\}$$

and μ on $\mathfrak{C}$ by

$$\mu(B) = \begin{cases} 0 & \text{if } B \cap A \text{ is countable,} \\ 1 & \text{if } B \cap A \text{ is cocountable in } A. \end{cases}$$

Suppose $\tilde{\mu}$ on $\mathfrak{B}$ extends μ. Observe that for any $B \in \mathfrak{B}$, $B \supset A$ we have $\mu(B) = 1$ so that $\tilde{\mu}^*(A) = 1$. Since A is a set of power $\aleph_1$ and since $\tilde{\mu}(\{x\}) = 0$ for every x, we have by $\mathrm{MA} + \aleph_1 < 2^{\aleph_0}$ that $\tilde{\mu}^*(A) = 0$ (see [4]). This contradiction shows that μ defined on $\mathfrak{C}$ has no extension to $\mathfrak{B}$.

COROLLARY 4. *The statement* "(b) $\Rightarrow$ (*)" *is undecidable in* ZFC.

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Reçu par la Rédaction le 15. 08. 1982