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FOURIER ANALYSIS OF THE BANACH INDICATRIX

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Let $\varphi(t)$ be a continuous function of bounded variation on $\langle a, b\rangle$, and let $N_\varphi(x)$ denote the number of solutions of the equation $\varphi(t) = x$, $t \in \langle a, b\rangle$; $N_\varphi(x)$ may be infinite. The function N_φ is known as the *Banach indicatrix* (cf. [4]). It was proved by Banach [1] that

$$\int_{-\infty}^{\infty} N_\varphi(y)\,dy = \int_a^b |d\varphi(t)| = \operatorname*{var}_{\langle a,b\rangle} \varphi. \tag{1}$$

We shall call the function φ *piece-wise monotonic* if there exists a finite partition $a = t_0 < t_1 < \ldots < t_n = b$ such that φ is monotonic (non-decreasing or non-increasing) in each interval $\langle t_{j-1}, t_j\rangle$.

Let φ be continuous and piece-wise monotonic and let $m_\varphi(x)$ denote the number of components of $\varphi^{-1}(x)$. Obviously, $0 \leqslant m_\varphi(x) < \infty$. Now define

$$N_\varphi^*(x) = \begin{cases} m_\varphi(x) & \text{if} \quad x \neq \varphi(a),\ x \neq \varphi(b), \\ m_\varphi(x) - \frac{1}{2} & \text{if} \quad x = \varphi(a) \neq \varphi(b) \text{ or } x = \varphi(b) \neq \varphi(a), \\ m_\varphi(x) - 1 & \text{if} \quad x = \varphi(a) = \varphi(b). \end{cases}$$

It is easy to see that $N_\varphi^*(x) \leqslant N_\varphi(x)$ for all $x \in (-\infty, \infty)$, and the equality holds except for at most a countable set of points. It was proved by Kac [3] that for piece-wise monotonic φ with continuous derivative the formula

$$N_\varphi^*(x) = \frac{1}{\pi} \int_0^\infty du \left[\int_a^b \cos u\big(\varphi(t) - x\big)\,|d\varphi(t)|\right] \tag{2}$$

holds for all real x. Kac established this formula without referring to the Banach indicatrix. It turns out, however, that by the methods of Fourier analysis a somewhat stronger result than (2) can be deduced from formula (1). We are also able to derive a similar formula for N_φ under the assumption that φ is continuous and of bounded variation on $\langle a, b\rangle$.

In order to state our results we need the definition of (C, k) summability for integrals. Let a be a continuous function on $\langle 0, \infty)$. Then we put $(k > -1)$ (cf. [2], p. 111)

$$(C, k)\int_0^\infty a(u)du = \lim_{T\to\infty}\int_0^T \left(1-\frac{u}{T}\right)^k a(u)du.$$

It is easy to see that

$$(C, 1)\int_0^\infty a(u)du = \lim_{T\to\infty}\frac{1}{T}\int_0^T \left[\int_0^\lambda a(u)du\right]d\lambda.$$

THEOREM 1. *Let φ be a continuous function of bounded variation on $\langle a, b\rangle$. Then*

$$N_\varphi(x) = (C, 1)\frac{1}{\pi}\int_0^\infty du\left[\int_a^b \cos u\big(\varphi(t)-x\big)|d\varphi(t)|\right]$$

for almost all real x. Moreover, the right-hand side of this equality converges to $\frac{1}{2}[N_\varphi(x_+)+N_\varphi(x_-)]$ at each point x, where the limits $N_\varphi(x_\pm)$ exist. The convergence is uniform over each finite and closed interval of continuity of N_φ.

Proof. Notice that the Banach result implies $N_\varphi \epsilon L^1(-\infty, \infty)$. Let $J = \langle c, d\rangle$, $-\infty < c < d < \infty$, and let $I_E(x)$ be 1 if $x \epsilon E$ and 0 if $x \notin E$. Applying (1) to the function

$$\varphi_J(t) = \max\big[c, \min\big(d, \varphi(t)\big)\big]$$

we obtain

$$\begin{aligned}(3)\qquad \int_{-\infty}^\infty N_{\varphi_J}(y)dy &= \int_{-\infty}^\infty I_J(y)N_\varphi(y)dy \\ &= \int_{\varphi^{-1}(J)} |d\varphi(t)| = \int_a^b I_J\big(\varphi(t)\big)|d\varphi(t)|.\end{aligned}$$

Now let E be a Borel subset of $(-\infty, \infty)$. Then using (3) we can show that

$$(4)\qquad \int_{-\infty}^\infty I_E(y)N_\varphi(y)dy = \int_a^b I_E\big(\varphi(t)\big)|d\varphi(t)|,$$

where the left-hand side is the Lebesgue integral and the right-hand side is the Lebesgue-Stieltjes integral. Formula (4) implies that

$$(5)\qquad \int_{-\infty}^\infty f(y)N_\varphi(y)dy = \int_a^b f\big(\varphi(t)\big)|d\varphi(t)|$$

holds for any bounded, real valued Borel function. In particular, if $f(y) = \cos u(x-y)$, where x and u are fixed real parameters, equation (5) gives

$$\int_{-\infty}^{\infty} N_\varphi(y)\cos u(y-x)\,dy = \int_a^b \cos u\bigl(\varphi(t)-x\bigr)\,|d\varphi(t)|. \tag{6}$$

The partial integral of the Fourier repeated integral of N_φ is

$$S_\omega(x) = \frac{1}{\pi}\int_0^\omega du\Bigl[\int_{-\infty}^{\infty} N_\varphi(y)\cos u(y-x)\,dy\Bigr],$$

hence by (6)

$$S_\omega(x) = \frac{1}{\pi}\int_0^\omega du\Bigl[\int_a^b \cos u\bigl(\varphi(t)-x\bigr)\,|d\varphi(t)|\Bigr]$$

To complete the proof it is sufficient to apply the (1·21) Theorem from [5], p. 246.

The suggestion was made to me by Lee Lorch to employ a Tauberian theorem to improve the summability in Theorem 1. This method leads to a slightly stronger result than that obtained by Kac.

THEOREM 2. *Let φ be a piece-wise monotonic and continuous function on $\langle a, b\rangle$. Then for each real x and for any $k > -1$ we have*

$$N_\varphi^*(x) = (C, k)\frac{1}{\pi}\int_0^\infty du\Bigl[\int_a^b \cos u\bigl(\varphi(t)-x\bigr)\,|d\varphi(t)|\Bigr].$$

In particular, the integral converges to $N_\varphi^*(x)$.

Proof. Let $I_x(y)$ be 0 if $y \neq x$ and 1 if $y = x$. Then by the very definition of N_φ^* we have

$$N_\varphi^*(x) = \frac{1}{2}\operatorname*{var}_{a\leqslant t\leqslant b} I_x\bigl(\varphi(t)\bigr).$$

The total variation of a given function of bounded variation is an additive function of intervals. Therefore

$$N_\varphi^*(x) = \tfrac{1}{2}\sum_{j=1}^{n}\operatorname*{var}_{\langle t_{j-1},t_j\rangle} I_x\bigl(\varphi(t)\bigr), \tag{7}$$

where $\langle t_{j-1}, t_j\rangle$, $j = 1, \ldots, n$, are the intervals of monotonicity of φ. Let

$$\psi_j(x) = \operatorname*{var}_{\langle t_{j-1},t_j\rangle} I_x\bigl(\varphi(t)\bigr), \quad j = 1, \ldots, n.$$

One checks that

$$\psi_j(x) \equiv 0 \quad \text{if} \quad \varphi(t_{j-1}) = \varphi(t_j),$$

and if $\varphi(t_j) \neq \varphi(t_{j-1})$, $a_j = \min[\varphi(t_{j-1}), \varphi(t_j)]$, $\beta_j = \max[\varphi(t_{j-1}), \varphi(t_j)]$, then

$$\psi_j(x) = \begin{cases} 2 & \text{for} \quad x \in (a_j, \beta_j), \\ 1 & \text{for} \quad x = a_j \text{ and } x = \beta_j, \\ 0 & \text{for} \quad x \notin \langle a_j, \beta_j \rangle. \end{cases}$$

Notice that

$$\psi_j(x) = \frac{\psi_j(x_+) + \psi_j(x_-)}{2} \quad \text{for} \quad x \in (-\infty, \infty), \tag{8}$$

and that

$$\operatorname*{var}_{(-\infty,\infty)} \psi_j(x) \leqslant 4 \quad \text{for} \quad j = 1, \ldots, n. \tag{9}$$

Combining (7), (8) and (9) we obtain

$$\operatorname*{var}_{(-\infty,\infty)} N_\varphi^*(x) \leqslant 2n < \infty, \tag{10}$$

and

$$N_\varphi^*(x) = \frac{N_\varphi^*(x_+) + N_\varphi^*(x_-)}{2} \quad \text{for} \quad x \in (-\infty, \infty). \tag{11}$$

We remember that $N_\varphi(y) = N_\varphi^*(y)$ for almost all y. Following step by step the proof of Theorem 1 it is not hard to see that the function N_φ, in Theorem 1, can be replaced by N_φ^*. This and (11) give

$$N_\varphi^*(x) = (C, 1) \frac{1}{\pi} \int_0^\infty du \Big[\int_a^b \cos u\big(\varphi(t) - x\big) |d\varphi(t)| \Big]. \tag{12}$$

Equation (6) implies

$$\begin{aligned} \int_a^b \cos u\big(\varphi(t) - x\big) |d\varphi(t)| &= \int_{-\infty}^{\infty} N_\varphi(y) \cos u(y - x)\, dy \\ &= \int_{-\infty}^{\infty} N_\varphi^*(y) \cos u(y - x)\, dy = \frac{1}{u} \int_{-\infty}^{\infty} N^*(y)\, d \sin u(y - x) \\ &= -\frac{1}{u} \int_{-\infty}^{\infty} \sin u(y - x)\, dN_\varphi^*(y), \end{aligned}$$

hence, by (10), for large u we get

$$\left|\int_a^b \cos u\big(\varphi(t)-x\big)\,|d\varphi(t)|\right| \leqslant \frac{2n}{u} = O(u^{-1}). \tag{13}$$

However, (12) and (13) are the hypotheses of the Tauberian theorem for integrals stated in §6.8 on p. 135 of [2]. Applying this theorem we get the required result.

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