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OVALOIDS WITH PRESCRIBED CURVATURE OF THE SECOND FUNDAMENTAL FORM

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1. An oriented connected surface M in the Euclidean three-space E^3 will be called *convex* if its second fundamental form II is positive definite. On a convex surface, II defines a Riemannian metric. Various questions arise in connection with the second Riemannian metric in M (see, e.g., the references in [1]). In particular, if M is closed (compact and without boundary) and convex (an ovaloid in short) we give two global characterizations of the sphere which generalize some results of [1], [2] and [5].

Let $\{e_1, e_2\}$ be any set of local frame fields around a point P and $\{w^1, w^2\}$ their dual one-forms. Let $\{w_j^i\}$, $i, j = 1, 2$, be the connection forms for the unique Riemannian connection of the metric $\langle , \rangle_I$ with respect to this basis, and $w_j'^i$ the connection forms for II with respect to the same basis. If

$$K_j^i = w_j'^i - w_j^i \quad \text{and} \quad K_j^i = \sum_{k=1,2} K_{jk}^i w^k,$$

then

$$(1) \qquad K_{II} = H - \frac{1}{K} \langle \operatorname{grad} H, X \rangle_I + Q \quad \text{and} \quad K_{II} = H - \frac{1}{8K^2} \nabla_{II} K + P,$$

where K_{II} is the Gauss curvature of II, K is the usual Gauss curvature of I, $H = (k_1 + k_2)/2$ is the mean curvature of the ovaloid, X is a vector field on M given by

$$X = \sum_{i,m=1,2} K_{mm}^i e_i,$$

and P and Q are certain nonnegative functions (see [4], p. 240, and [5]). If P_0 is a critical point of H or K, (1) yield

$$\frac{K_{II}(P_0)}{H(P_0)} \geqslant 1.$$

2. First we prove a theorem which generalizes Theorem 2.1 of [1].

THEOREM 2.1. *Let M be an ovaloid in E^3. If there exists a real function $f(x, y)$ which satisfies the following conditions:*

(a) *$f(x, y)$ is increasing in y,*

(b) *the function $G(x, y) = f(x, y)/\sqrt{y}$ $(y > 0)$ is decreasing in y,*

(c) *the function $h(x) = G(x, x^2)$ is monotonic,*

and if $K_{II} = f(H, K)$ identically on M, then M is a sphere.

Proof. Assume h to be increasing. We take P_0 such that

$$H(P_0) = \min_{P \in M} H(P).$$

Since $H^2 \geqslant K$, we get for any $P \in M$

$$\begin{aligned}\frac{K_{II}(P)}{\sqrt{K(P)}} &= \frac{f(H(P), K(P))}{\sqrt{K(P)}} = G(H(P), K(P)) \geqslant G(H(P), H^2(P)) \\ &= h(H(P)) \geqslant h(H(P_0)) = G(H(P_0), H^2(P_0)) \\ &= \frac{f(H(P_0), H^2(P_0))}{H(P_0)} \geqslant \frac{f(H(P_0), K(P_0))}{H(P_0)} = \frac{K_{II}(P_0)}{H(P_0)} \geqslant 1.\end{aligned}$$

Then $K_{II} \geqslant \sqrt{K}$ for any point of M, and so M is a sphere [3]. If h is decreasing, we take P_0 such that

$$H(P_0) = \max_{P \in M} H(P).$$

As an easy consequence of Theorem 2.1 we obtain

COROLLARY 2.1. *Let M be an ovaloid in E^3. If*

$$K_{II} = \sum_{i=1}^{n} c_i H^{s_i} K^{r_i},$$

where c_i, s_i, r_i are constants and $c_i > 0$, $0 \leqslant r_i \leqslant \frac{1}{2}$ and $s_i + 2r_i - 1 \geqslant 0$ (or $\leqslant 0$) for any $i = 1, 2, \ldots, n$, then M is a sphere.

Proof. The function

$$f(x, y) = \sum_{i=1}^{n} c_i x^{s_i} y^{r_i}$$

with c_i, s_i, r_i as above satisfies the conditions of Theorem 2.1, and so M is a sphere.

Remark. In particular, for $i = 1$, Corollary 2.1 is exactly Theorem 2.1 of [1].

The following theorem generalizes Theorems 1 and 1′ of [5].

THEOREM 2.2. *Let M be an ovaloid in E^3. If there exists a real function $f(x, y)$ which satisfies the following conditions:*

(a) *$f(x, y)$ is increasing in x,*

(b) *the function* $F(x, y) = f(x, y)/x$ $(x \neq 0)$ *is decreasing in* x,

(c) *the function* $g(y) = F(\sqrt{y}, y)$ $(y > 0)$ *is monotonic,*

and if $K_{II} = f(H, K)$ *identically on* M, *then* M *is a sphere.*

Proof. Assume g to be increasing. We take P_0 such that

$$K(P_0) = \min_{P \in M} K(P).$$

For any $P \in M$ we have

$$\frac{K_{II}(P)}{\sqrt{K(P)}} = \frac{f(H(P), K(P))}{\sqrt{K(P)}} \geqslant \frac{f(\sqrt{K(P)}, K(P))}{\sqrt{K(P)}}$$

$$= F(\sqrt{K(P)}, K(P)) = g(K(P)) \geqslant g(K(P_0)) = F(\sqrt{K(P_0)}, K(P_0))$$

$$\geqslant F(H(P_0), K(P_0)) = \frac{f(H(P_0), K(P_0))}{H(P_0)} = \frac{K_{II}(P_0)}{H(P_0)} \geqslant 1,$$

and so M is a sphere. If g is decreasing, we take P_0 such that

$$K(P_0) = \max_{P \in M} K(P).$$

Remark. Theorems 1 and 1′ of [5] are special cases of Theorem 2.2. For example (Theorem 1), if we assume that $f(x, y)$ is increasing in x, decreasing in y and $F(x, y)$ is decreasing in x, we must prove that $g(y)$ is monotonic. In fact, if $y_1 < y_2$, then

$$g(y_1) = F(\sqrt{y_1}, y_1) = \frac{f(\sqrt{y_1}, y_1)}{\sqrt{y_1}} \geqslant \frac{f(\sqrt{y_2}, y_1)}{\sqrt{y_2}}$$

$$\geqslant \frac{f(\sqrt{y_2}, y_2)}{\sqrt{y_2}} = F(\sqrt{y_2}, y_2) = g(y_2).$$

Also from Theorem 2.2 we obtain

COROLLARY 2.2. *Let* M *be an ovaloid in* E^3. *If*

$$K_{II} = \sum_{i=1}^{n} c_i H^{s_i} K^{r_i},$$

where c_i, s_i, r_i *are constants and* $c_i > 0$, $0 \leqslant s_i \leqslant 1$ *and* $s_i + 2r_i - 1 \geqslant 0$ (*or* $\leqslant 0$) *for any* $i = 1, 2, \ldots, n$, *then* M *is a sphere.*

Proof. The function

$$f(x, y) = \sum_{i=1}^{n} c_i x^{s_i} y^{r_i},$$

with c_i, s_i, r_i as above, satisfies the conditions of Theorem 2.2, and so M is a sphere.

Remark. In particular, for $i = 1$, Corollary 2.2 is exactly the Theorem of [2].

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