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THE n-MINIMAL CHROMATIC MULTIPLICITY OF A GRAPH

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For a given positive integer n and a graph G it may not be possible to express G as the edge sum of spanning n-chromatic subgraphs. For example, if $n > \chi(G)$, then G has no n-chromatic subgraphs and, clearly, no such factorization is possible. If G is nonempty and $n = 1$, then each subgraph must be empty and, certainly, no collection of empty graphs can have a nonempty edge sum. Even if we limit n to the range $2 \leqslant n \leqslant \chi(G)$, it is still possible that no such decomposition of G will exist. For example, let $G = K_4$ and $n = 3$.

It is true, however, that for any graph G and integer n ($2 \leqslant n \leqslant \chi(G)$) a factorization of G exists where all but one of the factors have chromatic number n and the one (possibly empty) exceptional factor has chromatic number less than n. We seek to minimize the number of factors in such a factorization. Our terminology follows that of [1].

More formally then, let G be a graph and let n be an integer such that $2 \leqslant n \leqslant \chi(G)$. The *n-minimal chromatic multiplicity* of G, denoted by $m(n, \chi, G)$, is the least positive integer k such that G can be expressed as the edge sum of spanning subgraphs $G_1, G_2, \ldots, G_k, R$, where $\chi(G_i) = n$ for $i = 1, 2, \ldots, k$ and $\chi(R) < n$.

Helpful related work has been done by Harary et al. [3]. For integers $n \geqslant 2$ they define the *n-particity* of a graph G, denoted by $\beta_n(G)$, to be the minimum number of factors in any factorization of G into n-partite spanning subgraphs. Earlier work on $\beta_2(G)$ was done by Harary [2], Nieminen [5], and Matula [4]. Our first result generalizes an argument presented in [3].

LEMMA. *If $\chi(H) = n^m$, then $\beta_n(H) \leqslant m$.*

Proof. We use induction on m. If $m = 1$, then $\chi(H) = n$ so that H is n-partite and $\beta_n(H) = 1$. Assume the result holds for $m = r$ and let J be a graph with $\chi(J) = n^{r+1}$. Let n^{r+1} colors used in coloring $V(J)$ be partitioned into n sets with n^r colors in each set. Denote these sets by $C_1, C_2, \ldots, C_n$. For $j = 1, 2, \ldots, n$ let J_j be the subgraph of J induced by those vertices of J

colored with any of the n^r colors of C_j. Clearly, $\chi(J_j) = n^r$ for $j = 1, 2, \dots, n$. By the inductive assumption, $\beta_n(J_j) \leqslant r$ for $j = 1, 2, \dots, n$.

For $q = 1, 2, \dots, n$ the set $E(J_q)$ may be partitioned into $\beta_n(J_q)$ subsets $E_1^q, E_2^q, \dots, E_{\beta_n(J_q)}^q$ with each subset inducing an n-partite graph in J_q. Note that $E_d^1 \cup E_d^2 \cup \dots \cup E_d^n$ induces an n-partite graph in J for $1 \leqslant d \leqslant \max\{\beta_n(J_q) \mid q = 1, 2, \dots, n\}$ since for such an integer d each of $\langle E_d^1 \rangle_J$, $\langle E_d^2 \rangle_J$, $\dots$, $\langle E_d^n \rangle_J$ is n-partite in J and has pairwise disjoint vertex sets.

Thus the graph I with vertex set $V(J)$ and edge set $E(J_1) \cup E(J_2) \cup \dots \cup E(J_n)$ satisfies

$$\beta_n(I) \leqslant \max\{\beta_n(J_q) \mid q = 1, 2, \dots, n\} \leqslant r.$$

The remaining edges of J, which are $E(J) - E(I)$, induce an n-partite subgraph of J with partite sets $V(J_1)$, $V(J_2)$, $\dots$, $V(J_n)$. Thus $\beta_n(J) \leqslant 1 + r$, which completes the proof.

The following notion is necessary for the presentation of our main result:

A graph G is said to have *lower n-particity* if there exists a factorization of G into $\beta_n(G)$ nonempty n-partite spanning subgraphs not all of which have chromatic number n. For example, consider the graph K_4. Note that $\beta_3(K_4) = 2$. Fig. 1 exhibits a factorization of K_4 into two nonempty 3-partite spanning subgraphs where one of the subgraphs does not have chromatic number 3. Thus K_4 has lower 3-particity. By contrast, $\beta_2(K_4) = 2$ and K_4 does not have lower 2-particity.

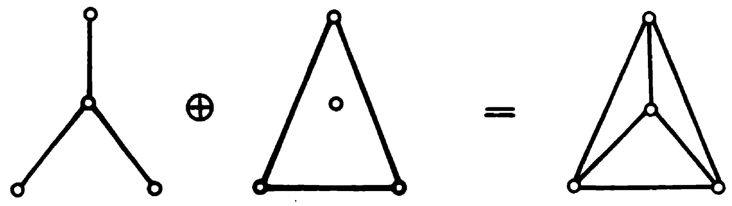

Fig. 1

THEOREM. *Let G be a graph and let n be an integer such that $2 \leqslant n \leqslant \chi(G)$. Then*

$$m(n, \chi, G) = \begin{cases} \{\log_n \chi(G)\} - 1 & \text{if } G \text{ has lower } n\text{-particity}, \\ \{\log_n \chi(G)\} & \text{otherwise}. \end{cases}$$

Proof. Let $m(n, \chi, G) = k$. Then G contains a collection of $k+1$ edge-disjoint spanning subgraphs $G_1, G_2, \dots, G_k, R$ whose edge sum is G and where $\chi(G_i) = n$ for $i = 1, 2, \dots, k$ and $\chi(R) < n$. Since all of these subgraphs are n-partite, we obtain

(1) $$\beta_n(G) \leqslant k+1.$$

Assume next that $\beta_n(G) = t$ and that $E(G) = E_1 \cup E_2 \cup \ldots \cup E_t$ is a partition of $E(G)$ where the graph F_i with vertex set $V(G)$ and edge set E_i is n-partite for $i = 1, 2, \ldots, t$. Thus the vertices of each subgraph F_i can be partitioned into n-partite sets. Now label the vertices of G where the label on each vertex v is a t-tuple $(s_1, s_2, \ldots, s_t)$ with $s_i = j$ if vertex v belongs to the j-th partite set when considered as a vertex of the graph F_i $(i = 1, 2, \ldots, t)$.

If two vertices $(s_1, s_2, \ldots, s_t)$ and $(s'_1, s'_2, \ldots, s'_t)$ are adjacent in G, then the edge joining them must belong to E_j for some subscript j such that $1 \leqslant j \leqslant t$. Hence these two vertices belong to different partite sets in the graph F_j implying that $s_j \neq s'_j$, and so

$$(s_1, s_2, \ldots, s_t) \neq (s'_1, s'_2, \ldots, s'_t).$$

This labeling of G is therefore a coloring of G. The set of all possible such labels has cardinality n^t so that $\chi(G) \leqslant n^t$ yielding $\log_n \chi(G) \leqslant t$. Since $\beta_n(G) = t$ is an integer, we obtain $\beta_n(G) \geqslant \{\log_n \chi(G)\}$, which when combined with (1) gives

$$m(n, \chi, G) \geqslant \{\log_n \chi(G)\} - 1. \tag{2}$$

Next, recall that F_i is n-partite for $i = 1, 2, \ldots, t$ so that $\chi(F_i) \leqslant n$ for $i = 1, 2, \ldots, t$. We now show that $m(n, \chi, G) \leqslant t$.

If $\chi(F_i) = n$ for each i such that $1 \leqslant i \leqslant t$, then we write

$$G = F_1 \oplus F_2 \oplus \ldots \oplus F_t \oplus R,$$

where R is the empty graph on $V(G)$ so that $m(n, \chi, G) \leqslant t$.

The only other possibility is that some graphs in the set $\{F_i \mid 1 \leqslant i \leqslant t\}$ have chromatic number less than n (i.e. G has lower n-particity). Without loss of generality we may assume that

$$n \geqslant \chi(F_1) \geqslant \chi(F_2) \geqslant \ldots \geqslant \chi(F_t).$$

Let s be the least positive integer with $\chi(F_i) < n$ for $s \leqslant i \leqslant t$. Let

$$H = F_s \oplus F_{s+1} \oplus \ldots \oplus F_t.$$

If $s = 1$ so that none of the F_i have chromatic number n, then edges may be removed from F_t (and from $F_{t-1}, F_{t-2}, \ldots, F_2$ if need be) and allocated to F_1 until this augmented F_1 does have chromatic number n. This augmentation is certainly possible since $\chi(G) \geqslant n$. Hence we may assume, at the onset, that $\chi(F_1) = n$. That is to say, that $s \geqslant 2$. Under this assumption we may write

$$G = F_1 \oplus F_2 \oplus \ldots \oplus F_{s-1} \oplus H.$$

If $\chi(H) < n$, then $m(n, \chi, G) \leqslant s-1 \leqslant t-1 < t$. If $\chi(H) \geqslant n$, then H contains an n-chromatic subgraph H_1, formed by augmenting F_s with edges from

$F_{s+1}, F_{s+2}, \ldots, F_t$ so that $H = H_1 \oplus H_2$, where $E(H_2) = E(H) - E(H_1)$, and thus

$$G = F_1 \oplus \ldots \oplus F_{s-1} \oplus H_1 \oplus H_2.$$

If $\chi(H_2) < n$, then $m(n, \chi, G) \leqslant s \leqslant t$. If $\chi(H_2) \geqslant n$, then H_2 contains an n-chromatic spanning subgraph.

This process may be continued and eventually we will write

$$G = F_1 \oplus F_2 \oplus \ldots \oplus F_{s-1} \oplus H_1 \oplus H_2 \oplus \ldots \oplus H_l \oplus H_{l+1}$$

with $\chi(H_{l+1}) < n$ and $\chi(H_i) = n$ for $i = 1, 2, \ldots, l$. Thus we have $m(n, \chi, G) \leqslant s - 1 + l$. Since

$$H_1 \oplus H_2 \oplus \ldots \oplus H_l \oplus H_{l+1} = H = F_s \oplus F_{s+1} \oplus \ldots \oplus F_t$$

and each of the n-chromatic graphs $H_1, H_2, \ldots, H_l$ was formed by augmenting a subgraph in the list $F_s, F_{s+1}, \ldots, F_t$, it follows that

$$l \leqslant t - s + 1 = |\{F_s, F_{s+1}, \ldots, F_t\}|.$$

Thus $s - 1 + l \leqslant t$ and

(3) $$m(n, \chi, G) \leqslant t = \beta_n(G).$$

Our next step is to show that $\beta_n(G) \leqslant \{\log_n \chi(G)\}$. Since $\chi(G) \geqslant n$, there exists a positive integer r with $n^{r-1} < \chi(G) \leqslant n^r$. Let J be a supergraph of G with $\chi(J) = n^r$. By the Lemma, $\beta_n(J) \leqslant r = \log_n \chi(J)$. Since G is a subgraph of J, $\beta_n(G) \leqslant \beta_n(J)$ yielding $\beta_n(G) \leqslant \log_n \chi(J) = r$. Now $n^{r-1} < \chi(G) \leqslant n^r$ implies $r - 1 < \log_n \chi(G) \leqslant r$, which forces $\{\log_n \chi(G)\} = r$. Hence

(4) $$\beta_n(G) \leqslant \{\log_n \chi(G)\}.$$

Combining inequalities (3) and (4) we obtain

(5) $$m(n, \chi, G) \leqslant \{\log_n \chi(G)\}.$$

Now, joining (2) and (5) we may write

$$\{\log_n \chi(G)\} - 1 \leqslant m(n, \chi, G) \leqslant \{\log_n \chi(G)\}.$$

Finally, we determine under what conditions $m(n, \chi, G)$ attains these bounds.

Assume first that G has lower n-particity. Then there exists a collection of $\beta_n(G) = t$ n-partite spanning subgraphs of G, say $G_1, G_2, \ldots, G_t$ with $G = G_1 \oplus G_2 \oplus \ldots \oplus G_t$, where not all of the G_i have chromatic number n. Without loss of generality we assume that $\chi(G_t) < n$. Applying an edge reallocation scheme similar to that used earlier in the proof, we obtain $m(n, \chi, G) \leqslant t - 1 = \beta_n(G) - 1$. Using (4) we have $m(n, \chi, G) \leqslant \{\log_n \chi(G)\} - 1$, which establishes $m(n, \chi, G) = \{\log_n \chi(G)\} - 1$ in view of (2).

Next assume that G does not have lower n-particity. Hence, for every collection of $\beta_n(G)$ n-partite edge-disjoint spanning subgraphs of G whose edge sum is G, each subgraph has chromatic number n. Recall that $m(n, \chi, G) = k$ and that $G = G_1 \oplus G_2 \oplus \ldots \oplus G_k \oplus R$ is a factorization of G with $\chi(G_i) = n$ for $i = 1, 2, \ldots, k$ and $\chi(R) < n$. Assume, for the moment, that $k < \beta_n(G)$. In our argument so far we have proved that $\beta_n(G) = \{\log_n \chi(G)\}$, so that $m(n, \chi, G) = k \geqslant \beta_n(G) - 1$ by (2). Thus $m(n, \chi, G) = \beta_n(G) - 1$ or, equivalently, $\beta_n(G) = k+1$. Now $\{G_1, G_2, \ldots, G_k, R\}$ is a collection of $\beta_n(G)$ edge-disjoint n-partite spanning subgraphs of G whose edge sum is G. It follows that $\chi(R) = n$, a contradiction. Thus our assumption that $k < \beta_n(G)$ is false and we may write $m(n, \chi, G) \geqslant \beta_n(G) = \{\log_n \chi(G)\}$. Using (5) we obtain

$$m(n, \chi, G) = \{\log_n \chi(G)\}.$$

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