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ON THE SEMILATTICE OF EXTENSIONS OF A PARTIAL ALGEBRA

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1. Introduction. In this note* we shall consider the extensions of a fixed partial algebra $(A, \boldsymbol{f}) = (A, (f_i)_{i\in I})$ of arbitrary type $\Delta = (K_i)_{i\in I}$. Our main result will give a characterization of the extensions of $(A, \boldsymbol{f})$ in terms of congruence relations in the universal completion $(\hat{A}, \hat{\boldsymbol{f}})$ of $(A, \boldsymbol{f})$. Some of the results can be interpreted directly as generalizations of Theorems 8 and 9 in [1] which deal only with special completions of a partial algebra. We will use the terminology and some of the statements of [3] and [2]. The unique closed homomorphic extension of the identity map $\mathrm{id}_A\colon (A, \boldsymbol{f}) \to (B, \boldsymbol{g})$ to an initial segment of $\hat{A}$ will be denoted by φ_B; its domain will be denoted by $\mathrm{dom}(B, \boldsymbol{g})$ or $\mathrm{dom}\, B$ and the congruence relation induced by φ_B in its domain by $\ker(B, \boldsymbol{g})$ or $\ker B$.

2. Characterizations of extensions. A partial algebra $(B, \boldsymbol{g})$ of type Δ is called an *extension* of algebra $(A, \boldsymbol{f})$ if $A \subset B$, if A generates B, i. e. $\complement_B A = B$, and if $f_i \subset g_i \cap (A^{K_i} \times A)$, for all $i \in I$, or simply $\boldsymbol{f} \subset \boldsymbol{g} \| A$. We will consider the following special extensions $(B, \boldsymbol{g})$ of $(A, \boldsymbol{f})$:

(1) $(B, \boldsymbol{g})$ is a *completion* of $(A, \boldsymbol{f})$ if $(B, \boldsymbol{g})$ is a complete algebra, i. e. $g_i\colon B^{K_i} \to B$ for all $i \in I$.

(2) $(B, \boldsymbol{g})$ is an *inner extension* of $(A, \boldsymbol{f})$ if $B = A$.

(3) $(B, \boldsymbol{g})$ is a *strong extension* of $(A, \boldsymbol{f})$ if $\boldsymbol{f} = \boldsymbol{g} \| A$.

(4) $(B, \boldsymbol{g})$ is an *initial extension* of $(A, \boldsymbol{f})$ if, for all $i \in I$ and $\boldsymbol{b} : K_i \to B$, $g_i(\boldsymbol{b}) \in A$ implies $\boldsymbol{b} : K_i \to A$.

Indeed, the defining condition of (4) just states that $(A, \boldsymbol{f})$ is an *initial segment* of its extension and (3) says that $(A, \boldsymbol{f})$ is a *relative algebra* of its extension. Since conditions (3) and (4) insure that the additional values

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of the extending operations g_i, $i \in I$, always are outside of $(A, \boldsymbol{f})$, we shall call such an extension an *exterior* extension of $(A, \boldsymbol{f})$; they were called normal extensions in [1]. It should be noted that the universal completion $(\hat{A}, \hat{\boldsymbol{f}})$ of $(A, \boldsymbol{f})$ is an exterior extension, and that (3) and (4), in fact, form one axiom of an inner characterization of $(\hat{A}, \hat{\boldsymbol{f}})$ (cf. [3]). Only exterior completions were considered in [1].

Some immediate consequences of the definitions above for an extension $(B, \boldsymbol{g})$ of $(A, \boldsymbol{f})$ are the following:

(5) $$\ker(B, \boldsymbol{g}) \cap (A \times A) = \mathrm{id}_A.$$

(6) If $(B, \boldsymbol{g})$ is a completion, then $\ker(\hat{A}, \hat{\boldsymbol{f}}) \subset \ker(B, \boldsymbol{g})$.

(7) If $(B, \boldsymbol{g})$ is an inner extension, then A is a complete system of representatives for $\ker(B, \boldsymbol{g})$.

THEOREM 1. *An extension $(B, \boldsymbol{g})$ of $(A, \boldsymbol{f})$ is strong iff*

(8) $$\ker(B, \boldsymbol{g}) \cap \big(A \times D(A)\big) = \mathrm{id}_A,$$

where, for any subset $X \subset \hat{A}$,

$$D(X) = X \cup \{\hat{f}_i(\boldsymbol{x}) \mid \boldsymbol{x}\colon K_i \to X,\ i \in I\}.$$

Since the identity map $\mathrm{id}_A\colon (A, \boldsymbol{f}) \to (B, \boldsymbol{g})$ is injective and strong if $(B, \boldsymbol{g})$ is a strong extension, this theorem is a restatement of the Corollary to Theorem 1 in [2].

THEOREM 2. *For an extension $(B, \boldsymbol{g})$ of $(A, \boldsymbol{f})$, the following statements are equivalent:*

(4) *$(B, \boldsymbol{g})$ is an initial extension.*

(9) *For every $y = \hat{f}\,(\boldsymbol{y}) \in \mathrm{dom}(B, \boldsymbol{g})$, if the class of y modulo $\ker(B, \boldsymbol{g})$ intersects A, then the class of $\boldsymbol{y}(k)$ modulo $\ker(B, \boldsymbol{g})$ intersects A, for all $k \in K_i$.*

(10) *$\varphi_B^{-1}(A)$ is an initial segment of $\hat{A}$.*

Proof. (4) $\Rightarrow$ (9). Let $y = \hat{f}_i(\boldsymbol{y}) \in \mathrm{dom}(B, \boldsymbol{g})$ and $x \in A$ such that

$$x = \varphi_A(x) = \varphi_B(y) = \varphi_B\big(\hat{f}_i(\boldsymbol{y})\big) = g_i(\varphi_B \circ \boldsymbol{y}).$$

Then there is an $\boldsymbol{a}\colon K_i \to A$ such that $\varphi_B \circ \boldsymbol{y} = \boldsymbol{a}$ since $(B, \boldsymbol{g})$ is an initial extension, i. e. the class of $\boldsymbol{y}(k)$, $k \in K_i$, modulo $\ker(B, \boldsymbol{g})$ intersects A.

(9) $\Rightarrow$ (10). Let $y = \hat{f}_i(\boldsymbol{y}) \in \varphi_B^{-1}(A) \subset \mathrm{dom}(B, \boldsymbol{g})$. Then there is an $x \in A$ such that $x = \varphi_B(y)$. By hypothesis, there exists a sequence $\boldsymbol{a}\colon K_i \to A$ such that $\boldsymbol{a} = \varphi_B \circ \boldsymbol{y}$, i. e. $\boldsymbol{y}$ is a sequence in $\varphi_B^{-1}(A)$. Therefore, $\varphi_B^{-1}(A)$ is an initial segment.

(10) $\Rightarrow$ (4). Suppose that $a = g_i(\boldsymbol{b}) \in A$ for some sequence $\boldsymbol{b}\colon K_i \to B$. Choose $\boldsymbol{y}\colon K_i \to \operatorname{dom}(B, \boldsymbol{g})$ such that $\boldsymbol{b} = \varphi_B \circ \boldsymbol{y}$. Then $a = g_i(\boldsymbol{b}) = g_i(\varphi_B \circ \boldsymbol{y}) = \varphi_B(\hat{f}_i(\boldsymbol{y}))$, i.e. $\hat{f}_i(\boldsymbol{y}) \in \varphi_B^{-1}(A)$. By our assumption, $\boldsymbol{y}\colon K_i \to \varphi_B^{-1}(A)$ and, therefore, $\boldsymbol{b} = \varphi_B \circ \boldsymbol{y}\colon K_i \to A$, proving that $(B, \boldsymbol{g})$ is an initial extension.

Using the characterizations of the preceding theorems we obtain a simple description for exterior extensions.

THEOREM 3. *$(B, \boldsymbol{g})$ is an exterior extension of $(A, \boldsymbol{f})$ iff*

$$\ker(B, \boldsymbol{g}) \cap (A \times \operatorname{dom}(B, \boldsymbol{g})) = \mathrm{id}_A. \tag{11}$$

Proof. We prove the first part by algebraic induction. Let $(B, \boldsymbol{g})$ be an exterior extension. Obviously, the inclusion "$\supset$" is trivial. So let $(x, y) \in \ker(B, \boldsymbol{g})$ with $x \in A$ and $y \in \operatorname{dom}(B, \boldsymbol{g})$. If $y \in A$, then $x = y$ by (5). Assume now, for $y = \hat{f}_i(\boldsymbol{y}) \in \operatorname{dom}(B, \boldsymbol{g})$ and $\boldsymbol{y} : K_i \to \operatorname{dom}(B, \boldsymbol{g})$, that $\boldsymbol{y}(k)$ satisfies the assertion for all $k \in K_i$; i. e. whenever

$$(\boldsymbol{a}(k), \boldsymbol{y}(k)) \in \ker(B, \boldsymbol{g}) \cap (A \times \operatorname{dom}(B, \boldsymbol{g})) \quad \text{for some } \boldsymbol{a}(k),\ k \in K_i,$$

then $\boldsymbol{a}(k) = \boldsymbol{y}(k)$.

By our choice of y, the class of y modulo $\ker(B, \boldsymbol{g})$ intersects A. Therefore, from Theorem 2, we obtain a sequence $\boldsymbol{x}\colon K_i \to A$ such that $\boldsymbol{x} = \varphi_B \circ \boldsymbol{y}$ and, by induction hypothesis, we conclude $\boldsymbol{x} = \boldsymbol{y}$. Finally, $(x, \hat{f}_i(\boldsymbol{x})) = (x, y) \in \ker(B, \boldsymbol{g}) \cap (A \times D(A))$, so that, by Theorem 1, $x = y$.

Conversely, assume that equality (11) holds and that $a = g_i(\boldsymbol{b}) \in A$ for some $\boldsymbol{b}\colon K_i \to B$. Choose a sequence $\boldsymbol{y}\colon K_i \to \operatorname{dom}(B, \boldsymbol{g})$ such that $\boldsymbol{b} = \varphi_B \circ \boldsymbol{y}$; then $(a, \hat{f}_i(\boldsymbol{y}))$ belongs to $\ker(B, \boldsymbol{g}) \cap (A \times \operatorname{dom}(B, \boldsymbol{g})) = \mathrm{id}_A$. Since $(\hat{A}, \hat{\boldsymbol{f}})$ is an exterior extension, $a = \hat{f}_i(\boldsymbol{y})$ implies $\boldsymbol{y}\colon K_i \to A$ and $a = f_i(\boldsymbol{y})$ and, therefore, $\boldsymbol{b} = \boldsymbol{y}$ and $g_i(\boldsymbol{b}) = f_i(\boldsymbol{y})$.

As a special case of Theorem 3, we get the following result of [1]:

COROLLARY. *Let $(B, \boldsymbol{g})$ be a completion of $(A, \boldsymbol{f})$. Then $(B, \boldsymbol{g})$ is an exterior completion iff* $\ker(B, \boldsymbol{g}) \cap (A \times \hat{A}) = \mathrm{id}_A$.

3. The semilattice of extensions. The previous theorem shows that the exterior extensions of a partial algebra are the "nice" extensions. In fact, Theorem 3 states that the congruence relation $\ker(B, \boldsymbol{g})$ splits into the disjoint union of the congruence id_A in A and some congruence in $\operatorname{dom}(B, \boldsymbol{g}) - A$. Arbitrary intersections and joins of such congruence relations are of the same type so that the congruences $\ker(B, \boldsymbol{g})$ of exterior extensions $(B, \boldsymbol{g})$ form a complete lattice with smallest element id_A, corresponding to $(A, \boldsymbol{f})$, and largest element $\mathrm{id}_A \cup \{(\hat{A} - A) \times (\hat{A} - A)\}$, corresponding to "the" one-point completion $(\dot{A}, \dot{\boldsymbol{f}})$ which is defined by $\dot{A} = A \,\dot{\cup}\, \{\infty\}$ and, for $i \in I$, $\dot{f}_i \| A = f_i$ and $\dot{f}_i(\boldsymbol{a}) = \infty$ if $\boldsymbol{a} \notin \operatorname{dom} f_i$.

The situation becomes much more complex when one passes to arbitrary extensions, i. e. those that may or may not satisfy (3) or (4). An extension (completion) may be an alternating iteration of extensions of type (2) and of type (3) or (4) as the following example shows.

Consider an algebra $(A_1, f_1) = (\{1, 2\}, \varphi)$ with an empty binary operation.

Define extensions (A_n, f_n) of (A_1, f_1) recursively by putting

$$A_{n+1} = A_n \cup \{2n+1, 2n+2\}, \quad f_{n+1} \| A_n = f_n,$$

$$(+) \qquad f_{n+1}(i, j) = \min\{i, j\} + 2 \quad \text{if } 2n-1 \leqslant i, j \leqslant 2n,$$

$$(++) \qquad f_{n+1}(i, j) = \min\{i, j\}$$

if either $i \leqslant 2n$ or $j \leqslant 2n$ (but not both), and $(i, j) \notin \operatorname{dom} f_n$.

The table of values of f_{n+1} then has the following form:

f_{n+1}	1	2	3	4	f_2 5	...	$2n+1$	f_{n+1} $2n+2$
1	3	3	1	1	1	....	1	1
2	3	4	2	2	2	....	2	2
3	1	2	5	5	3	....	3	3
4	1	2	5	6	4		4	4
5	1	2	3	4			5	5
⋮	⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮
$2n+1$	1	2	3	4	5	....	$2n+3$	$2n+3$
$2n+2$	1	2	3	4	5	....	$2n+3$	$2n+4$

It is easily seen that

$$\operatorname{dom} f_n = (A_n \times A_n) - (\{2n-1, 2n\} \times \{2n-1, 2n\}).$$

Therefore, each (A_n, f_n), $n \in N$, is a proper extension of (A_{n-1}, f_{n-1}), where f_n is not defined on four-tuples. Also, these extensions form a chain the union of which is a completion of algebra (A_1, f_1). Finally, at each stage n, the extension $(A_{n+1}, f_{n+1}^{(+)})$ of (A_n, f_n) — where only the $(+)$-part of the definition of f_{n+1} is used — is an exterior extension of (A_n, f_n), whereas (A_{n+1}, f_{n+1}) is an inner extension of $(A_{n+1}, f_{n+1}^{(+)})$ such that $A_1 \subset \subset \operatorname{im} f_{n+1}$.

This example shows that it is impossible to characterize an arbitrary extension by a split of its corresponding congruence into a "lower" part — for inner extension — and an "upper" part — for extensions of type (3) and/or (4). In general, these two parts may alternate infinitely often; in particular, none of the two parts can be characterized by an initial segment of $\hat{A}$. Despite this freedom in choosing the extension as exemplified above, it is possible to characterize the congruence relations $\ker(B, \boldsymbol{g})$ of arbitrary extensions $(B, \boldsymbol{g})$.

THEOREM 4. *Suppose we are given a pair* (R, Q) *with the following properties*:

(E1) *R is a closed congruence relation in an initial segment* $\mathrm{dom}\,R$ *of* $\hat{A}$ *with* $\mathrm{id}_A \subset R$.

(E2) *Q is a union of equivalence classes of R.*

(E3) *A is a complete system of representatives for the congruence* $R\cap(Q\times Q) =: R_Q$.

(E4) $R\cap\big(Q\times D(Q)\big) = R\cap(Q\times Q)$.

Then $(B, \boldsymbol{g}) := (\mathrm{dom}\,R, \hat{\boldsymbol{f}}\|\mathrm{dom}\,R)/R$ *is an extension of* $(A, \boldsymbol{f})$ *and* $(A, \boldsymbol{g}\|A) \cong (Q, \hat{\boldsymbol{f}}\|Q)/R_Q$.

Proof. Consider the following diagram:

$$\begin{tikzcd}
(A, \boldsymbol{f}) \arrow[r, "\mathrm{id}_A"] \arrow[dr, dashed, "j_1"'] & (Q, \hat{\boldsymbol{f}}\|Q) \arrow[r, "\mathrm{id}_Q"] \arrow[d, "q"] & (\mathrm{dom}\ R, \hat{\boldsymbol{f}}\|\mathrm{dom}\ R) \arrow[d, "p"] \\
 & (Q/R_Q, \hat{\boldsymbol{f}}^Q) := (Q, \hat{\boldsymbol{f}}\|Q)/R_Q \arrow[r, dashed, "j_2"'] & (B, \boldsymbol{g})
\end{tikzcd}$$

where p is the natural projection, q — its restriction to Q, and j_1 and j_2 the induced homomorphisms. Note that the existence and injectivity of j_1 follows from (E2) and (E3). Since q is the restriction of p and since the image of $p\circ\mathrm{id}_Q$ generates B, we may apply the Homomorphism Theorem ([3], Theorem 7) to the right cell of the diagram. Therefore, j_2 exists and is an injective homomorphism, so that the whole diagram commutes. The set A generates $\mathrm{dom}\,R$; hence A generates B, i. e. $(B, \boldsymbol{g})$ is an extension of $(A, \boldsymbol{f})$. Property (E3) implies, in particular, that j_1 is a bijective homomorphism, i. e. that $(Q/R_Q, \hat{\boldsymbol{f}}^Q)$ is an inner extension of $(A, \boldsymbol{f})$. In order to show that it is a relative algebra of $(B, \boldsymbol{g})$ let us identify its underlying set with the set A. Assume that $\boldsymbol{a}\colon K_i \to A$ such that $a = g_i(\boldsymbol{a})\in A$. Then there is an $\boldsymbol{x}\colon K_i \to Q$ such that $\boldsymbol{a} = q\circ\boldsymbol{x}$. Choose $x\in Q$ with $q(x) = a$. Then $\big(x, \hat{f}_i(\boldsymbol{x})\big)\in R\cap\big(Q\times D(Q)\big)$ implies by (E4) that $\hat{f}_i(\boldsymbol{x})\in Q$, and we obtain $q\big(\hat{f}_i(\boldsymbol{x})\big) = \hat{f}_i^Q(q\circ\boldsymbol{x}) = \hat{f}_i^Q(\boldsymbol{a})$. Therefore, $\hat{\boldsymbol{f}}^Q$ is the relative structure of $(B, \boldsymbol{g})$ on A.

The following theorem establishes a converse to Theorem 4:

THEOREM 5. *Let* $(B, \boldsymbol{g})$ *be an extension of* $(A, \boldsymbol{f})$. *Then the pair* $\big(\ker(B, \boldsymbol{g}), \varphi_B^{-1}(A, \boldsymbol{g}\|A)\big)$ *satisfies the four axioms* (E1)-(E4).

Proof. By definition of the pair $\big(\ker(B, \boldsymbol{g}), \varphi_B^{-1}(A)\big)$, conditions (E1)-(E3) are fulfilled. In order to verify (E4), let $(x, y)\in\ker(B, \boldsymbol{g})$ such that $x \in \varphi_B^{-1}(A)$ and $y = \hat{f}_i(\boldsymbol{y})$ with $\boldsymbol{y}\colon K_i \to \varphi_B^{-1}(A)$. Then we get $\varphi_B(x) = \varphi_B(y) = \varphi_B(\hat{f}_i(\boldsymbol{y})) = g_i(\varphi_B \circ \boldsymbol{y}) \in A$ which implies $\hat{f}_i(\boldsymbol{y}) \in \varphi_B^{-1}(A)$. This shows the non-trivial inclusion of the identity (E4).

Define now $\boldsymbol{E}(A) := \{(R, Q) \mid R$ and Q satisfy (E1)-(E4)$\}$ and define a partial order on this set by $(R, Q) \supset (R', Q')$ if and only if $R \supset R'$ and $Q \supset Q'$. To this partial order corresponds the (dual) quasi-order on the class of all extensions of $(A, \boldsymbol{f})$ defined by $(B, \boldsymbol{g}) \rhd (B', \boldsymbol{g}')$ if and only if there exists a unique homomorphism $\varphi_{BB'}\colon (B, \boldsymbol{g}) \to (B', \boldsymbol{g}')$ such that $\varphi_{BB'}|A = \mathrm{id}_A$. This homomorphism $\varphi_{BB'}$ need not be a surjective map; but its image generates B' since $(B', \boldsymbol{g}')$ is generated by $(A, \boldsymbol{f})$. The equivalence relation $\sim$ associated with this quasi-order $\rhd$, i. e. $(B, \boldsymbol{g}) \sim (B', \boldsymbol{g}')$ iff $(B, \boldsymbol{g}) \rhd (B', \boldsymbol{g}') \rhd (B, \boldsymbol{g})$, determines precisely the isomorphism classes of extensions of $(A, \boldsymbol{f})$. From our characterization of extensions by pairs (R, Q) satisfying (E1)-(E4) we get

THEOREM 6. *The partially ordered set* $(\boldsymbol{E}(A), \supset)$ *"is" the (dual) partial ordering of the isomorphism classes of all extensions of algebra* $(A, \boldsymbol{f})$.

THEOREM 7. $(\boldsymbol{E}(A), \supset)$ *is an infimum-semilattice with smallest element* (id_A, A) *such that the meet of every non-empty subset exists.*

Proof. Let (R_t, Q_t), $t \in T$, be a non-empty family of elements of $\boldsymbol{E}(A)$. Define $R := \bigcap\{R_t \mid t \in T\}$ and $Q := \bigcup\{R(a) \mid a \in A\}$, where $R(a)$ denotes the equivalence class of $a \in A$ modulo R. First we verify that the pair (R, Q) belongs to $\boldsymbol{E}(A)$. The axioms (E1) to (E3) are satisfied by definition. For (E4), let $(x, y) \in R \cap (Q \times \mathrm{dom} R)$. Since (E3) holds for (R, Q), there is an element $a \in A$ such that $(a, x) \in R$. Therefore, $(a, y) \in R$ and we obtain $y \in R(a) \subset Q$. In fact, we have shown more than the non-trivial inclusion of (E4), i.e. $R \cap (Q \times D(Q)) \subset R \cap (Q \times \mathrm{dom} R) \subset R \cap (Q \times Q)$. From the definition of the pair (R, Q) it follows immediately that (R, Q) is a lower bound for the given family (R_t, Q_t), $t \in T$. In order to see that it is the greatest lower bound consider the following diagram:

$$\begin{tikzcd}
 & (B, \boldsymbol{g}) \arrow[r, "\mathrm{id}_B"] & (X, \boldsymbol{k}) \arrow[dd, "p_t"] \\
(A, \boldsymbol{f}) \arrow[ur, "j"] \arrow[dr, "\mathrm{id}_A"'] & & \\
 & (B', \boldsymbol{g}') \arrow[uu, dashed, "\varphi"] \arrow[uur, dashed, "\varphi"'] \arrow[r, "\varphi_t"'] & (B_t, \boldsymbol{g}_t)
\end{tikzcd}$$

where $(B_t, \boldsymbol{g}_t)$, $t \in T$, are extensions of $(A, \boldsymbol{f})$ corresponding to the given family (R_t, Q_t), $t \in T$, and where $(X, \boldsymbol{k})$ is the cartesian product of $(B_t, \boldsymbol{g}_t)$, $t \in T$, with its natural projections p_t. With our construction of (R, Q) we may realize (R, Q) as a subalgebra $(B, \boldsymbol{g})$ of $(X, \boldsymbol{k})$ such that $(A, \boldsymbol{f})$ is identified via the embedding j with the diagonal of $(X, \boldsymbol{k})$; $\mathrm{id}_B \circ j$ is the lifting of the family $\mathrm{id}_A\colon (A, \boldsymbol{f}) \to (B_t, \boldsymbol{g}_t)$. Let $(B', \boldsymbol{g}')$ be an extension of $(A, \boldsymbol{f})$ corresponding to a lower bound (R', Q') of the family (R_t, Q_t),

$t \in T$, in $\boldsymbol{E}(A)$; i. e. $(B', \boldsymbol{g}')$ is an upper bound for $(B_t, \boldsymbol{g}_t)$, $t \in T$, in the quasi-order $\triangleright$. Finally, the homomorphisms φ_t satisfy the equalities $\varphi_t|A = \mathrm{id}_A$ for all $t \in T$. If φ is the induced homomorphism into the product $(X, \boldsymbol{k})$, then $\varphi \circ \mathrm{id}_A = \mathrm{id}_B \circ j$ and φ maps, therefore, A onto the diagonal of X. Since $(B, \boldsymbol{g})$ is generated by $(A, \boldsymbol{f})$, the image of φ is contained in $(B, \boldsymbol{g})$, so that $(B', \boldsymbol{g}') \triangleright (B, \boldsymbol{g})$, i.e. $(R', Q') \subset (R, Q)$. Hence (R, Q) is the greatest lower bound of the family (R_t, Q_t), $t \in T$.

As the proof of Theorem 7 shows, the meet in the semilattice $(\boldsymbol{E}(A), \supset)$ is the set-theoretical intersection only for the first component of its elements, but not for the second component, in general. Using the notations of the proof, the following inclusion is used implicitly in the proof:

$$\begin{aligned}(12)\qquad Q &= \bigcup\{R(a) \mid a \in A\} = \bigcup\left\{\bigcap\{R_t(a) \mid t \in T\} \mid a \in A\right\} \\ &\subset \bigcap\left\{\bigcup\{R_t(a) \mid a \in A\} \mid t \in T\right\} = \bigcap\{Q_t \mid t \in T\}.\end{aligned}$$

Equality in (12) is forced trivially if, for any $s, t \in T$, $R_s \cap (Q_s \times Q_s) = R_t \cap (Q_t \times Q_t)$. Interpreting this identity for the corresponding extensions, we have $(A, \boldsymbol{g}_s\|A) = (A, \boldsymbol{g}_t\|A)$ since Q_t together with $R_t\|Q_t$, $t \in T$, describes just the inner extension of $(A, \boldsymbol{f})$ which is contained in extension $(B_t, \boldsymbol{g}_t)$.

COROLLARY 1. $\boldsymbol{E}(A; R, Q) := \{(R', Q') \mid R'\|Q' = R\|Q\} \subset \boldsymbol{E}(A)$ *is a subsemilattice of* $\boldsymbol{E}(A)$ *for each* $(R, Q) \in \boldsymbol{E}(A)$. *The infimum in* $\boldsymbol{E}(A; R, Q)$ *is the set-theoretical intersection.*

It should be noted that apart from above remarks still nothing can be said beyond Corollary 1 if no conditions are put on Q. Denote by $Q\downarrow$ the initial segment generated by Q, i. e. the smallest subset D of $\hat{A}$ containing Q such that for $x \in D$ and $x = \hat{f}_i(\boldsymbol{x})$ also $\boldsymbol{x}(k) \in D$ for all $k \in K_i$. Then for each (R', Q') in $\boldsymbol{E}(A; R, Q)$, $Q' = Q \subset \mathrm{dom} R'$ and R' is determined on Q, but not on $Q\downarrow - Q$. The extensions on p. 196 and simple modifications thereof provide examples. In the special case that Q itself is an initial segment of $\hat{A}$, condition (E4) implies that R_Q is a closed congruence relation in Q. Therefore, $(R_Q, Q) \in \boldsymbol{E}(A)$ and it is, in fact, the smallest element of $\boldsymbol{E}(A; R, Q)$.

Another special subsemilattice of $\boldsymbol{E}(A)$ is described in

COROLLARY 2. $\boldsymbol{V}(A) := \{(R, Q) \mid \mathrm{dom} R = \hat{A}\} \subset \boldsymbol{E}(A)$ *is a subsemilattice of* $\boldsymbol{E}(A)$ *with smallest element* $(\mathrm{id}_{\hat{A}}, A)$, *corresponding to the universal completion* $(\hat{A}, \hat{\boldsymbol{f}})$ *of* $(A, \boldsymbol{f})$.

It is obvious that neither $\boldsymbol{E}(A)$ nor its subsemilattices described in the corollaries have a greatest element, in general. The semilattice $\boldsymbol{V}(A)$ contains some of the maximal elements in $\boldsymbol{E}(A)$ since $\mathrm{dom} R = \hat{\boldsymbol{A}}$ is a necessary condition for a maximal (R, Q). Naturally, if for such a pair (R, Q) also $Q = \hat{A}$, then (R, Q) is a maximal element of $\boldsymbol{E}(A)$.

THEOREM 8. *$E(A)$ has a greatest element if and only if $|A| = 1$ or (A, f) is a complete algebra.*

Proof. $\Rightarrow$. If $E(A)$ has a greatest element (R, Q), let (B, g) be the corresponding extension of (A, f). Then (B, g) is a completion of (A, f) and a homomorphic image of every completion of (A, f), in particular of every inner completion. Therefore, $B = A$ and g is the only complete structure on set A extending f, so that for $|A| > 1$, f is already a complete structure.

$\Leftarrow$. If $|A| = 1$, then (A, f) admits only one inner completion which is a homomorphic image of every completion of (A, f). If (A, f) is a complete algebra, then $E(A) = \{(A, f)\}$.

4. An open problem. In the previous section we described some of the more elementary properties of the set $E(A)$ of extensions of algebra (A, f). These investigations originated with the problem that we want to state in this section. Let $\mathscr{A} := \mathrm{HSP}(A, f)$, i. e. the variety generated by algebra (A, f), and let $\mathscr{A}_V$ be the subvariety of all complete algebras in $\mathscr{A}$. Obviously, $\mathscr{A}_V$ is a non-trivial class whenever $|A| > 1$. On the other hand, put $\mathscr{V} := \{(A, g) | (A, g) \text{ inner completion of } (A, f)\}$ and let $\mathscr{V}_A = \mathrm{HSP}(\mathscr{V})$, the variety generated by $\mathscr{V}$.

PROBLEM. Characterize those algebras (A, f) for which $\mathscr{A}_V = \mathscr{V}_A$ is true. (**P 896**)

It is always true that $\mathscr{V}_A \subset \mathscr{A}_V$ since every inner completion of (A, f) is a homomorphic image of (A, f). The converse is false, in general, as the following example shows.

Consider algebras with a unary operation and put

$$\mathscr{A} = \mathrm{HSP}(A, f) = \mathrm{HSP}(\{0, 1\}, \varphi).$$

Then there are four inner completions of (A, f). Since $\mathscr{A}$ is closed under S and P, the set of natural numbers N is a subalgebra (i. e. subset) of some power of (A, f). Therefore, $(N, ')$ with the successor operation as algebraic structure belongs to $\mathscr{A}_V$. Suppose now $(N, ') \in \mathscr{V}_A$. Then there is a surjective homomorphism $p\colon (B, g) \to (N, ')$ where (B, g) is a subalgebra of some product (X, k) of the four inner completions of (A, f). Choose $b_0 \in B$ such that $p(b_0) = 0 \in N$. Since $(N, ')$ is a Peano-algebra, there is a homomorphism $j\colon (N, ') \to (B, g)$ such that $j(0) = b_0$. Therefore, $p \circ j = \mathrm{id}_N$, i. e. j is injective and we may consider $(N, ')$ to be embedded into (X, k) as a subalgebra. Let $0 \in N$ be represented by the sequence $(a_i)_{i \in I} \in X$ with $a_i \in \{0, 1\}$, and let $2 = k(k(0))$. Then $4 = k^2(2) = 2$ since of the four complete operations on A three act like the identity on 2 and since the fourth has period two. This clearly is a contradiction, so that $(N, ') \notin \mathscr{V}_A$.

It is easy to see that $|A| = 1$ or (A, f) a complete algebra are sufficient to force $\mathscr{A}_V = \mathscr{V}_A$. Unfortunately, they are not necessary for the equality as they are in Theorem 8. Again, consider algebras with one unary operation. This time, take $A = \boldsymbol{N}$ and, for some $n \in \boldsymbol{N}$, let f be the successor operation defined up to this number n. Then $(\boldsymbol{N}, ')$ itself is an inner extension of (A, f), and since for our special type, the free algebra over a set M is given by the disjoint union of M copies of $(\boldsymbol{N}, ')$, we obtain the equality $\mathscr{A}_V = \mathscr{V}_A$.

A solution of the Problem might be possible if one can find an internal characterization of the infimum in $\boldsymbol{E}(A)$ of all inner completions of (A, f).

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