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THE DOMAIN OF ATTRACTION OF NON-GAUSSIAN STABLE DISTRIBUTION IN A HILBERT SPACE, II

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The full characterization of measures attracted by non-Gaussian stable distributions in a separable real Hilbert space can be found in papers [3] and [5]. In [5] the problem is investigated for full stable measures. However, a weaker assumption, namely the assumption of infinite dimension of the stable measure, enables a simpler description of its domain of attraction. The aim of this paper is to prove the above fact.

Let H be an infinite-dimensional, separable, real Hilbert space and let $\boldsymbol{B}(H)$ be the family of Borel subsets of H. Denote by $\mathfrak{M}$ the set of all probability distributions in H with the weak convergence topology.

The least set closed in H such that the measure of its complement equals zero is said to be the *support* of the finite measure μ defined on $\boldsymbol{B}(H)$. The dimension of the closed linear hull of the support of μ is called the *dimension of* μ.

The distribution concentrated at a point $x \in H$ will be denoted by δ_x and the n-th convolution power of the distribution p by p^{n*}.

The symbol $T_a p$, where $a > 0$ and $p \in \mathfrak{M}$, will stand for the following distribution:

$$T_a p(A) = p\{x \in H: ax \in A\} \quad \text{for every } A \in \boldsymbol{B}(H).$$

A distribution $q \in \mathfrak{M}$ is said to be *stable* if for any positive numbers a and b there exist both a positive number c and an element $x \in H$ such that

$$T_a q * T_b q = T_c q * \delta_x.$$

It can be shown that a distribution is stable if and only if it is the limit distribution for a sequence of distributions of the form $T_{a_n} p^{n*} * \delta_{x_n}$, where $a_n > 0$, $x_n \in H$, and $p \in \mathfrak{M}$ (cf. [2]).

The set of all distributions $p \in \mathfrak{M}$ for which there exist sequences $\{a_n\}$ and $\{x_n\}$ such that the sequence of distributions $T_{a_n} p^{n*} * \delta_{x_n}$ is weakly convergent to a stable distribution q is called the *domain of attraction of the distribution* q.

In [2] it has been shown that the distribution q is a stable measure in H if and only if

(a) q is a Gaussian distribution in H, or

(b) the characteristic functional of q is of the form

$$\hat{q}(y) = \exp\left\{i(x_0, y) + \int_{H-\{\theta\}} \left[e^{i(x,y)} - 1 - \frac{i(x, y)}{1+\|x\|^2}\right] M(dx)\right\},$$

where $x_0 \in H$, and M is a σ-finite measure in H which is finite on the complement of every neighbourhood of zero in H and satisfies the following conditions:

$$\int_{\|x\|\leqslant 1} \|x\|^2 M(dx) < +\infty \tag{1}$$

and there exists a $\lambda \in (0, 2)$ such that

$$T_a M = a^\lambda M \quad \text{for every } a > 0. \tag{2}$$

The number λ is called the *type of the non-Gaussian stable distribution q.*

LEMMA. *A non-Gaussian stable distribution q is infinite dimensional if and only if its corresponding measure M satisfies*

(3) $M\{x \in H: \|\pi_N x\| \geqslant 1\} > 0$ *for any natural* N,

where $\pi_N x = \sum_{i=N}^{\infty} (x, e_i) e_i$ *and* $\{e_i\}$ *is a basis in* H.

Proof. We may assume that $x_0 = 0$ in (b). Suppose that, for some natural N_0,

$$M\{x \in H: \|\pi_{N_0} x\| \geqslant 1\} = 0.$$

By (2), $M\{x \in H: \|\pi_{N_0} x\| \geqslant \varepsilon\} = 0$ for every $\varepsilon > 0$, i. e. $\hat{q}(\pi_{N_0} y) = 1$ for every $y \in H$. Hence

$$q\{x \in H: (x, \pi_{N_0} y) = 0\} = 1 \quad \text{for every } y \in H,$$

and since H is separable, we have

$$q\{x \in H: \pi_{N_0} x = \theta\} = 1,$$

that is, the dimension of q is less than or equal to $N_0 - 1$.

Let us assign to a distribution $p \in \mathfrak{M}$ satisfying

$$\int_H \|x\|^2 p(dx) = +\infty \tag{4}$$

a family of measures M_X defined on $\boldsymbol{B}(H)$ by

$$M_X(B) = \frac{p\{XB\}}{p\{x \in H: \|x\| \geqslant X\}} \quad \text{for every } B \in \boldsymbol{B}(H). \tag{5}$$

Let M^{ε} stand for a measure M reduced to the set $\{x \in H: \|x\| > \varepsilon\}$.

THEOREM 1. *A distribution $p \in \mathfrak{M}$ lies in the attraction domain of some infinite-dimensional stable distribution of the type $\lambda \in (0, 2)$ if and only if*

(I) $$\lim_{X\to+\infty} \frac{p\{x \in H: \|x\| \geqslant X\}}{p\{x \in H: \|x\| \geqslant kX\}} = k^{\lambda} \text{ for every } k > 0,$$

(II) *for any $\varepsilon > 0$, measures $(M_X)^{\varepsilon}$ are, as $X \to +\infty$, weakly convergent to M^{ε}, where M is a measure in H satisfying* (3).

Proof. Note that (I) implies (4) and thus measures M_X can be defined. Conditions (I) and (II) are necessary by the Theorem in [3] and by the Lemma.

Suppose now that (I) and (II) are satisfied. It is proved in [3] that M has property (2). It also has property (1). Indeed, (I) implies (cf. [5], p. 157)

(6) $$\lim_{X\to+\infty} \frac{\int_{\|x\|\leqslant X} \|x\|^2 p(dx)}{X^2 p\{x \in H: \|x\| \geqslant X\}} = \frac{\lambda}{2-\lambda}.$$

Therefore, by (II), (5), and (6),

$$\int_{\varepsilon<\|x\|\leqslant 1} \|x\|^2 M(dx) = \lim_{X\to+\infty} \frac{\int_{\varepsilon X\leqslant\|x\|\leqslant X} \|x\|^2 p(dx)}{X^2 p\{x \in H: \|x\| \geqslant X\}} \leqslant \frac{\lambda}{2-\lambda}$$

for any $\varepsilon > 0$.

We prove condition (iii) of the Theorem in [3]. The condition may be written in the form

(iii) $$\lim_{N\to\infty} \overline{\lim_{X\to+\infty}} \frac{\int_{\|x\|\leqslant X} \|\pi_N x\|^2 p(dx)}{\int_{\|x\|\leqslant X} \|x\|^2 p(dx)} = 0.$$

Consider the set $B_N = \{x \in H: \|\pi_N x\| \geqslant 1\}$. By (2), B_N is the continuity set of M. Thus

(7) $$\lim_{X\to+\infty} M_X(B_N) = \lim_{X\to+\infty} \frac{p\{x \in H: \|\pi_N x\| \geqslant X\}}{p\{x \in H: \|x\| \geqslant X\}} = M(B_N) > 0.$$

Then, by (I) and (7), we have

(8) $$\lim_{X\to+\infty} \frac{p\{x \in H: \|\pi_N x\| \geqslant X\}}{p\{x \in H: \|\pi_N x\| \geqslant kX\}} = k^{\lambda}$$

for any $k > 0$ and any natural N. Condition (8) obviously implies

(9) $$\lim_{X\to+\infty} \frac{\int_{\|\pi_N x\|\leqslant X} \|\pi_N x\|^2 p(dx)}{X^2 p\{x \in H: \|\pi_N x\| \geqslant X\}} = \frac{\lambda}{2-\lambda}.$$

Condition (iii) follows now from (6), (9), (7), and from the fact that

$$\lim_{N\to\infty} M(B_N) = 0.$$

Therefore, it suffices to apply the Theorem in [3].

The characteristic functional of the stable measure of the type $\lambda \in (0, 2)$ in H may be rewritten in the form

$$(\mathrm{b}_1) \qquad \hat{q}(y) = \exp\left\{i(x_0, y) + \int_S \int_0^\infty \left[e^{ir(s,y)} - 1 - \frac{ir(s, y)}{1+r^2}\right] \frac{dr}{r^{1+\lambda}} \Gamma(ds)\right\},$$

where $x_0 \in H$, $S = \{x \in H: \|x\| = 1\}$, and Γ is a finite Borel measure in S (see [4]).

Define a measure γ on the positive half-line as follows:

$$(10) \qquad \gamma(A) = \int_A \frac{dr}{r^{1+\lambda}} \quad \text{for any } A \in \boldsymbol{B}(R_+).$$

If M is a measure in H corresponding to the stable distribution q according to (b) and Γ is a measure in S associated with the form (b_1) of the distribution q, then

$$(11) \qquad M(A) = \gamma \times \Gamma\left\{(r, s): r = \|x\|, s = \frac{x}{\|x\|}, x \in A\right\} \quad \text{for every } A \in \boldsymbol{B}(H).$$

This follows from the fact that the least σ-field containing all sets of the form

$$\left\{x \in H: \|x\| \in B, \frac{x}{\|x\|} \in W\right\}, \quad \text{where } B \in \boldsymbol{B}(R_+),\ W \in \boldsymbol{B}(S),$$

contains $\boldsymbol{B}(H)$.

By (11) we have

$$M\{x \in H: \|\pi_N x\| \geqslant 1\} = \int_1^\infty \Gamma\left\{s \in S: \|\pi_N s\| \geqslant \frac{1}{r}\right\} \frac{1}{r^{1+\lambda}} dr,$$

and thus the following condition is equivalent to (3):

(3_1) for any natural N there exists a number $\varepsilon \in (0, 1)$ such that $\Gamma\{s \in S: \|\pi_N s\| \geqslant \varepsilon\} > 0$.

Let us assign to a distribution $p \in \mathfrak{M}$ satisfying (4) the family of measures Γ_X defined on $\boldsymbol{B}(S)$ by

$$(12) \qquad \Gamma_X(W) = \lambda \frac{p\{x \in H: \|x\| \geqslant X, x/\|x\| \in W\}}{p\{x \in H: \|x\| \geqslant X\}} \quad \text{for every } W \in \boldsymbol{B}(S).$$

We shall characterize distributions attracted by the infinite-dimensional stable distribution of the type $\lambda \in (0, 2)$ in terms of the measures Γ_X.

THEOREM 2. *The distribution $p \in \mathfrak{M}$ lies in the attraction domain of some infinite-dimensional stable distribution of the type $\lambda \in (0, 2)$ if and only if condition* (I) *of Theorem* 1 *is satisfied and, moreover,*

(II_1) *the measures Γ_X are, as $X \to +\infty$, weakly convergent to a finite measure Γ in S satisfying condition* (3_1).

Proof. By (5) and (12) we have

$$(13)\quad M_X\left\{x \in H: \|x\| \geqslant r,\ \frac{x}{\|x\|} \in W\right\} = \frac{1}{\lambda}\Gamma_{rX}(W)\frac{p\{x \in H: \|x\| \geqslant rX\}}{p\{x \in H: \|x\| \geqslant X\}}$$

for every set $W \in \boldsymbol{B}(S)$. Putting $r = 1$ in (13) and applying Theorem 1 we show the necessity of the conditions.

To prove the sufficiency we shall show that condition (II) of Theorem 1 with the limit measure M defined by (11) is satisfied. Let $\varepsilon > 0$. Relation (13) implies

$$(14)\quad \lim_{X \to +\infty} (M_X)^\varepsilon(A) = M^\varepsilon(A)$$

for any set

$$A = \{x \in H: \|x\| \in B,\ x/\|x\| \in W\},$$

where $B \in \boldsymbol{B}(R_+)$, $W \in \boldsymbol{B}(S)$, and W is the continuity set of Γ.

Thus, to prove (II) we shall show weak compactness of the $(M_X)^\varepsilon$. Consider the measures on the plane defined as follows:

$$Q^\varepsilon_{N,X}(B) = M_X\left\{x \in H: \|x\| > \varepsilon, \left[\|x\|, \frac{\|\pi_N x\|}{\|x\|}\right] \in B\right\} \quad \text{for any } B \in \boldsymbol{B}(R^2_+),$$

$$Q^\varepsilon_N(B) = M\left\{x \in H: \|x\| > \varepsilon, \left[\|x\|, \frac{\|\pi_N x\|}{\|x\|}\right] \in B\right\} \quad \text{for any } B \in \boldsymbol{B}(R^2_+).$$

By (14) we have

$$(15)\quad \lim_{X \to +\infty} Q^\varepsilon_{N,X}(B) = Q^\varepsilon_N(B)$$

for any set $B = B_1 \dot{\times} B_2$, where $B_1, B_2 \in \boldsymbol{B}(R_+)$, and $\{s \in S: \|\pi_N s\| \in B_2\}$ is the continuity set of Γ. Hence (15) is satisfied for every $B \in \boldsymbol{B}(R^2_+)$ provided that it is the continuity set of measure Q^ε_N. Replace in (15) the set B by the set $\{(u, v) \in R^2_+: u \cdot v \geqslant 1\}$ and let $\varepsilon \in (0, 1)$. We get

$$\lim_{X \to +\infty} M_X\{x \in H: \|\pi_N x\| \geqslant 1\} = M\{x \in H: \|\pi_N x\| \geqslant 1\}.$$

Therefore, using (3_1) we obtain the condition identical with (7), i.e.

$$\lim_{X \to +\infty} \frac{p\{x \in H: \|\pi_N x\| \geqslant X\}}{p\{x \in H: \|x\| \geqslant X\}} = M\{x \in H: \|\pi_N x\| \geqslant 1\} > 0.$$

Similarly as in the proof of Theorem 1, condition (7) implies (iii) of Theorem in [3].

To show now that the measures $(M_X)^\varepsilon$ are weakly compact for any $\varepsilon > 0$ it suffices to use the conditions for weak compactness of measures in H given, for example, in [1], p. 448. Those conditions can be derived from (I) and (iii).

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