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ON CONVEX HYPERSURFACES IN E^{n+1}

BY

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1. Introduction. A convex hypersurface M in E^{n+1} with positive principal curvatures has a positive definite second fundamental form if it is appropriately oriented. Also its third fundamental form III is a positive definite metric on M with Gaussian curvature equal to 1. We denote by $\mathrm{I} = g_{ij}\,dx^i\,dx^j$, $\mathrm{II} = b_{ij}\,dx^i\,dx^j$, and $\mathrm{III} = e_{ij}\,dx^i\,dx^j$ the first, second, and third fundamental forms of M, respectively, and write $g = \det(g_{ij})$, $b = \det(b_{ij})$, and $e = \det(e_{ij})$. It is well known that

$$M_n = \frac{b}{g} = \sqrt{\frac{e}{g}}. \tag{1.1}$$

If we denote by Γ_{ij}^k, Π_{ij}^k, and Λ_{ij}^k the Christoffel symbols with respect to I, II, and III, then using a well-known result (cf. [2], p. 33) we conclude that in every case the functions

$$T_{ij}^k = \Gamma_{ij}^k - \Pi_{ij}^k, \tag{1.2}$$

$$\bar{T}_{ij}^k = \Lambda_{ij}^k - \Pi_{ij}^k, \tag{1.3}$$

$$S_{ij}^k = \Lambda_{ij}^k - \Gamma_{ij}^k \tag{1.4}$$

are components of tensors on M. In a similar way ([3], p. 22-23) we can prove the relations

$$T_{ij}^k = -\tfrac{1}{2} b^{rk}\,{}_{\mathrm{I}}\nabla_r b_{ij}, \tag{1.5}$$

$$\bar{T}_{ij}^k = -\tfrac{1}{2} b^{rk}\,{}_{\mathrm{III}}\nabla_r b_{ij}, \tag{1.6}$$

$$T_{ij}^k + \bar{T}_{ij}^k = 0, \tag{1.7}$$

where b^{ij} is the inverse matrix of b_{ij} and ${}_{\mathrm{I}}\nabla_r$, ${}_{\mathrm{III}}\nabla_r$ are the symbols of covariant differentiation with respect to I and III, respectively.

It is well known (cf. [1], p. 67) that the l-th mean curvature M_l of a hypersurface M is defined by

$$\binom{n}{l} M_l = \sum k_1 k_2 \ldots k_l,$$

where k_i $(i = 1, 2, \ldots, n)$ are the principal curvatures of M. It is also well known that on a convex hypersurface we have $M_1 M_{l-1} \geqslant M_l$ with equality in the case of the hypersphere.

In the sequel we assume that all the hypersurfaces have positive principal curvatures.

2. Main results. First we prove the following

LEMMA 2.1. *Let* $\mathrm{I} = g_{ij} dx^i dx^j$ *and* $\mathrm{II} = \bar{g}_{ij} dx^i dx^j$ *be two arbitrary Riemannian metrics on a hypersurface* M *(not necessarily closed) in* E^{n+1}. *Let* ${}_{\mathrm{I}}\Gamma^k_{ij}$ *and* ${}_{\mathrm{II}}\Gamma^k_{ij}$ *be the Christoffel symbols with respect to* I *and* II, *respectively, and* ${}_{\mathrm{II}}\nabla_l$ *the covariant differentiation with respect to* II. *If for raising and lowering the indices we use the tensor* $\bar{g}_{ij}$, *then*

$$(2.1)\qquad {}_{\mathrm{II}}\nabla_l A^{lj}_j - {}_{\mathrm{II}}\nabla_j A^{lj}_l = R_{\mathrm{II}} - \bar{g}^{il}\,{}_{\mathrm{I}}R_{il} + A^{lh}_l A^j_{hj} - P_{\mathrm{II}},$$

where $A^k_{ij} = {}_{\mathrm{I}}\Gamma^k_{ij} - {}_{\mathrm{II}}\Gamma^k_{ij}$, R_{II} *is the scalar curvature of* II, $P_{\mathrm{II}} = A^k_{ij} A^{ij}_k$ *is a function on* M, ${}_{\mathrm{I}}R_{il}$ *is the Ricci tensor of* I, *and* $\bar{g}^{il}$ *is the inverse matrix of* $\bar{g}_{il}$.

Proof. A direct computation gives ([2], p. 33)

$${}_{\mathrm{II}}\nabla_l A^k_{ij} - {}_{\mathrm{II}}\nabla_j A^k_{il} = {}_{\mathrm{I}}R^k_{ijl} - {}_{\mathrm{II}}R^k_{ijl} + A^k_{hj} A^h_{il} - A^k_{hl} A^h_{ij},$$

where ${}_{\mathrm{I}}R^k_{ijl}$ and ${}_{\mathrm{II}}R^k_{ijl}$ are the components of curvature tensors of I and II, respectively. Contracting once and transvecting with $\bar{g}^{il}$ we obtain (2.1).

Now, we can prove the following theorems.

THEOREM 2.1. *Let* M *be a convex hypersurface (not necessarily closed) in* E^{n+1}. *If* Δ_{I} *is the Laplace operator with respect to* I, *and* S *is the square of the length of* II, *then*

$$(2.2)\qquad \Delta_{\mathrm{I}} \log M_n - \mathrm{div}_{\mathrm{I}}\,\lambda = n^2 M_1^2 - nS + \frac{n}{M_n} \nabla_{\mathrm{II}}(M_1, M_n) - P_{\mathrm{I}},$$

where ∇_{II} *denotes the first Beltrami operator with respect to the second fundamental form* II *of* M, M_l $(l = 1, 2, \ldots, n)$ *is the* l*-th mean curvature of* M, λ *is a vector field with components*

$$\lambda^i = n b^{ir} \frac{\partial M_1}{\partial x^r},$$

and P_{I} *is a nonnegative function on* M.

Proof. Applying Lemma 2.1 in the case of the first and third fundamental forms of a convex hypersurface M in E^{n+1} we get

$$(2.3)\qquad {}_{\mathrm{I}}\nabla_l S^{lj}_j - {}_{\mathrm{I}}\nabla_j S^{lj}_l = R_{\mathrm{I}} - g^{il}\,{}_{\mathrm{III}}R_{il} + S^{lh}_l S^j_{hj} - P_{\mathrm{I}},$$

where R_{I} is the scalar curvature of I, and ${}_{\mathrm{III}}R_{il}$ is the Ricci tensor of III. But ${}_{\mathrm{III}}R_{il} = (n-1)e_{il}$, and thus $g^{il}\,{}_{\mathrm{III}}R_{il} = (n-1) g^{il} e_{il} = (n-1)S$ since $g^{il} e_{il} = S$. Also $R_{\mathrm{I}} = n^2 M_1^2 - S$ (cf. [1], p. 55). Moreover,

$$S^i_{ij} = A^i_{ij} - \Gamma^i_{ij} = \frac{\partial \log M_n}{\partial x^j} = \frac{1}{M_n} \frac{\partial M_n}{\partial x^j},$$

and thus

$$S_j^{lj} = g^{lm} S_{mj}^j = g^{lm} \frac{\partial \log M_n}{\partial x^m}.$$

Using (1.2)-(1.7) we get $S_{ij}^k = -2T_{ij}^k$ and, consequently,

$$S_j^{jl} = g^{jm} S_{mj}^l = g^{jm}(-2T_{mj}^l) = g^{jm} b^{lr} {}_{\mathrm{I}}\nabla_r b_{mj}$$

by (1.5), or

$$S_j^{jl} = b^{lr} {}_{\mathrm{I}}\nabla_r (g^{jm} b_{mj}) = n b^{lr} \frac{\partial M_1}{\partial x^r}$$

since $g^{jm} b_{mj} = nM_1$.

Finally, we have

$$S_l^{lh} S_{hj}^j = n b^{hr} \frac{\partial M_1}{\partial x^r} \frac{1}{M_n} \frac{\partial M_n}{\partial x^h} = \frac{n}{M_n} b^{hr} \frac{\partial M_1}{\partial x^r} \frac{\partial M_n}{\partial x^h} \quad \text{or} \quad S_l^{lh} S_{hj}^j = \frac{n}{M_n} \nabla_{\mathrm{II}}(M_1, M_n).$$

Setting λ^i as in the theorem and substituting the above relations in (2.3) we get

$${}_{\mathrm{I}}\nabla_l \left(g^{lm} \frac{\partial \log M_n}{\partial x^m} \right) - {}_{\mathrm{I}}\nabla_j \lambda^j = n^2 M_1^2 - nS + \frac{n}{M_n} \nabla_{\mathrm{II}}(M_1, M_n) - P_{\mathrm{I}}$$

or

$$g^{lm} {}_{\mathrm{I}}\nabla_{lm} \log M_n - \mathrm{div}_{\mathrm{I}}\, \lambda = n^2 M_1^2 - nS + \frac{n}{M_n} \nabla_{\mathrm{II}}(M_1, M_n) - P_{\mathrm{I}}$$

or (2.2). This completes the proof of Theorem 2.1.

THEOREM 2.2. *Let M be a convex hypersurface (not necessarily closed) in E^{n+1}. If Δ_{III} is the Laplace operator with respect to* III, *then*

(2.4)

$$-\Delta_{\mathrm{III}} \log M_n - n \,\mathrm{div}_{\mathrm{III}}\, \mu = \frac{n^2(M_n - M_1 M_{n-1})}{M_n} - \frac{n}{M_n} \nabla_{\mathrm{II}}\left(\frac{M_{n-1}}{M_n}, M_n \right) - P_{\mathrm{III}},$$

where μ is a vector field with components

$$\mu^i = b^{ir} \frac{\partial (M_{n-1}/M_n)}{\partial x^r}$$

and P_{III} is a nonnegative function on M.

Proof. Applying Lemma 2.1 in the case of the third and first fundamental forms of M we obtain

(2.5) $$ {}_{\mathrm{III}}\nabla_j S_l^{lj} - {}_{\mathrm{III}}\nabla_l S_j^{lj} = R_{\mathrm{III}} - e^{il} {}_{\mathrm{I}}R_{il} + S_l^{lh} S_{hj}^j - P_{\mathrm{III}},$$

where $R_{\rm III}$ is the scalar curvature of III, and e^{ij} is the inverse matrix of e_{ij}. But $R_{\rm II} = n(n-1)$ and ${}_{\rm I}R_{il} = nM_1 b_{il} - e_{il}$. Moreover,

$$S^j_{ij} = \frac{\partial \log M_n}{\partial x^i} = \frac{1}{M_n}\frac{\partial M_n}{\partial x^i}, \qquad S^{lj}_j = e^{lm} S^j_{mj} = e^{lm}\frac{\partial \log M_n}{\partial x^m}.$$

Also, using (1.2)-(1.7) we get $S^k_{ij} = 2\bar{T}^k_{ij}$, and thus

$$S^{jl}_j = e^{jm} S^l_{mj} = 2e^{jm}(-\tfrac{1}{2}b^{lr}{}_{\rm III}\nabla_r b_{mj})$$

by (1.5), or

$$S^{jl}_j = -b^{lr}{}_{\rm III}\nabla_r(e^{jm} b_{mj}) = -nb^{lr}\frac{\partial(M_{n-1}/M_n)}{\partial x^r}$$

since $e^{jm} b_{jm} = nM_{n-1}/M_n$.

Finally, we have

$$S^{lh}_l S^j_{hj} = -nb^{hr}\frac{\partial(M_{n-1}/M_n)}{\partial x^r}\frac{1}{M_n}\frac{\partial M_n}{\partial x^h} = -\frac{n}{M_n}b^{hr}\frac{\partial(M_{n-1}/M_n)}{\partial x^r}\frac{\partial M_n}{\partial x^h}$$

or

$$S^{lh}_l S^j_{hj} = -\frac{n}{M_n}\nabla_{\rm II}\left(\frac{M_{n-1}}{M_n}, M_n\right).$$

Setting μ^i as in the theorem and substituting the above relations in (2.5) we get

$$
\begin{aligned}
&{}_{\rm III}\nabla_j(-n\mu^j) - {}_{\rm III}\nabla_l\left(e^{lm}\frac{\partial \log M_n}{\partial x^m}\right) \\
&\qquad = \frac{n^2(M_n - M_1 M_{n-1})}{M_n} - \frac{n}{M_n}\nabla_{\rm II}\left(\frac{M_{n-1}}{M_n}, M_n\right) - P_{\rm III}
\end{aligned}
$$

or (2.4).

Now, we are ready to prove the main result of this paper.

THEOREM 2.3. *Let M be an ovaloid in E^{n+1}. Then*

$$\int_M \frac{\nabla_{\rm II}(M_1, M_n)}{M_n} dM \geq 0 \quad \textit{and} \quad \int_M \nabla_{\rm II}\left(\frac{M_{n-1}}{M_n}, M_n\right) dM \leq 0.$$

The equality in either case holds iff M is a hypersphere.

Proof. Using the divergence theorem of Stokes we infer from (2.2) that

$$\int_M (nS - n^2 M_1^2 + P_{\rm I})\, dM = n\int_M \frac{\nabla_{\rm II}(M_1, M_n)}{M_n} dM. \tag{2.6}$$

But $R_{\rm I} = n^2 M_1^2 - S$ ([1], p. 55). It is also obvious that $R_{\rm I} = n(n-1)M_2$. Since $M_1^2 \geq M_2$, we get $n^2 M_1^2 - S = R_{\rm I} = n(n-1)M_2 \leq n(n-1)M_1^2$ or

$S \geqslant nM_1^2$, and thus $nS - n^2 M_1^2 + P_1 \geqslant 0$. From this relation and (2.5) we obtain

$$\int_M \frac{\nabla_{\mathrm{II}}(M_1, M_n)}{M_n} dM \geqslant 0. \tag{2.7}$$

In (2.7) the equality holds iff M is a hypersphere. In fact, if M is a hypersphere, then $\nabla_{\mathrm{II}}(M_1, M_n) = 0$, and thus

$$\int_M \frac{\nabla_{\mathrm{II}}(M_1, M_n)}{M_n} dM = 0.$$

Conversely, if the last equality holds, then by (2.6) we obtain $nS - n^2 M_1^2 + P_1 = 0$, since the function $nS - n^2 M_1^2 + P_1$ is nonnegative, or $S = nM_1^2$. Then $M_2 = M_1^2$ or M is a hypersphere.

Moreover, from (2.5) by the Stokes theorem we get

$$\int_M \left(\frac{n^2(M_1 M_{n-1} - M_n)}{M_n} + P_{\mathrm{III}}\right) dM_{\mathrm{III}} = -n \int_M \frac{\nabla_{\mathrm{II}}(M_{n-1}/M_n)}{M_n} dM_{\mathrm{III}},$$

where dM_{III} is the volume element of the third fundamental form. But $dM_{\mathrm{III}} = M_n dM$, and thus

$$\int_M (n^2(M_1 M_{n-1} - M_n) + P_{\mathrm{III}} M_n) dM = -n \int_M \nabla_{\mathrm{II}}\left(\frac{M_{n-1}}{M_n}, M_n\right) dM. \tag{2.8}$$

Since $M_1 M_{n-1} - M_n \geqslant 0$, we have

$$\int_M \nabla_{\mathrm{II}}\left(\frac{M_{n-1}}{M_n}, M_n\right) dM \geqslant 0$$

with equality in the case of the hypersphere. In fact, if M is a hypersphere, then

$$\nabla_{\mathrm{II}}\left(\frac{M_{n-1}}{M_n}, M_n\right) = 0 \quad \text{and} \quad \int_M \nabla_{\mathrm{II}}\left(\frac{M_{n-1}}{M_n}, M_n\right) dM = 0.$$

Conversely, if the latter equality holds, then from (2.8) we conclude that $n^2(M_1 M_{n-1} - M_n) + M_n P_{\mathrm{III}} = 0$ or $M_1 M_{n-1} - M_n = 0$. The last equation proves that M is a hypersphere. This completes the proof of the theorem.

As an easy consequence of Theorem 2.3 we obtain

COROLLARY 2.1. *Let M be an ovaloid in E^{n+1}. If one of the functions M_1, M_n, and M_{n-1}/M_n is constant, then M is a hypersphere.*

Remark. Obviously, Theorem 2.3 gives a characterization of the hypersphere and is some generalization of known characterizations of the hypersphere which were given by some authors.

An immediate result of Theorem 2.3 is the following

COROLLARY 2.2. *Let M be an ovaloid in E^{n+1}. If there exists a function $\Phi\colon R\times R\to R$ which is increasing (decreasing) in one variable and strictly decreasing (strictly increasing) in the other variable and if*

$$\Phi\left(\frac{M_{n-1}}{M_n}(p),\ M_n(p)\right)=0 \quad \textit{for all } p\in M,$$

then M is a hypersphere.

Proof. If x_i $(i=1, 2, \ldots, n)$ are the line curvature coordinates, then the second fundamental form II of M takes the form

$$\mathrm{II}=\sum_{i=1}^{n} L_i (dx_i)^2$$

(L_i are positive since the principal curvatures are positive by assumption), and thus

$$\nabla_{\mathrm{II}}\left(\frac{M_{n-1}}{M_n},\ M_n\right)=\sum_{i=1}^{n}\frac{1}{L_i}\left(\frac{M_{n-1}}{M_n}\right)_{x_i}(M_n)_{x_i}. \tag{2.9}$$

But from the equality $\Phi(M_{n-1}/M_n,\ M_n)=0$ we get

$$\Phi_{M_{n-1}/M_n}\left(\frac{M_{n-1}}{M_n}\right)_{x_i}+\Phi_{M_n}(M_n)_{x_i}=0 \tag{2.10}$$

or

$$(M_n)_{x_i}=-\frac{\Phi_{M_{n-1}/M_n}}{\Phi_{M_n}}\left(\frac{M_{n-1}}{M_n}\right)_{x_i}$$

if we assume that Φ is strictly increasing or decreasing in the second variable (so $\Phi_{M_n}\neq 0$). Then from (2.9) and (2.10) we obtain

$$\nabla_{\mathrm{II}}\left(\frac{M_{n-1}}{M_n},\ M_n\right)=-\frac{\Phi_{M_{n-1}/M_n}}{\Phi_{M_n}}\sum_{i=1}^{n}\frac{1}{L_i}\left((M_n)_{x_i}\right)^2.$$

Since $\Phi_{M_{n-1}/M_n}\Phi_{M_n}\leqslant 0$ by assumptions, we have

$$\nabla_{\mathrm{II}}\left(\frac{M_{n-1}}{M_n},\ M_n\right)\geqslant 0 \quad \text{or} \quad \int_M \nabla_{\mathrm{II}}\left(\frac{M_{n-1}}{M_n},\ M_n\right)dM\geqslant 0. \tag{2.11}$$

Using Theorem 2.2 and (2.11) we conclude that M is a hypersphere.

In a similar way we obtain the following corollary which was proved in [4] by a complicated method.

COROLLARY 2.3. *Let M be an ovaloid in E^{n+1}. If there exists a function $\Phi\colon R\times R\to R$ which is increasing or decreasing in both variables and strictly monotonic in at least one of its variables and if $\Phi(M_1(p),\ M_n(p))=0$ for all $p\in M$, then M is a hypersphere.*

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