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ON SOME RANDOM CONVEX SETS

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1. For a sequence $\{z_k\}$ of complex numbers we denote by V_k the smallest closed convex set containing all elements of the sequence $\{z_k, z_{k+1}, \ldots\}$. The set $\bigcap_{k=1}^{\infty} V_k$, which we denote by $\operatorname{core}_k z_k$, is called the *core* of the sequence $\{z_k\}$ (cf. [2], p. 55). In analogy to the theory of linear prediction $\operatorname{core}_k z_k$ may be treated as the "convex remote future of the sequence $\{z_k\}$".

Let $\xi_1, \xi_2, \ldots$ be a sequence of random variables with complex values over a probability space $(\Omega, \mathscr{L}, P)$. If for every $\omega \in \Omega$ we put

$$V(\omega) = \operatorname{core}_\nu \xi_\nu(\omega), \tag{1}$$

then we obtain a family of convex sets associated with the process $\{\xi_\nu\}$. For example, if $(\eta_k\}$ is a sequence of independent random variables uniformly distributed on the interval $[0, 1]$, and $\xi_k = \exp(ik\eta_k)$, then $\operatorname{core}_\nu \xi_\nu = \{|z| \leqslant 1\}$ with the probability one.

Write (1) in a somewhat different form. Let Z denote the set of all complex numbers. For $z \in Z$ and $\omega \in \Omega$ we put

$$\eta(z, \omega) = \begin{cases} 1 & \text{if } z \in V(\omega), \\ 0 & \text{if } z \notin V(\omega). \end{cases} \tag{2}$$

THEOREM 1. *For every $z \in Z$ the function $\eta(z, .)$ is $\mathscr{L}$-measurable, i.e.,* (2) *describes a stochastic process* (*with "complex time z"*).

Proof. For $0 \leqslant a < \pi$ we set

$$V_a(\omega) = \{r \text{ real} \mid re^{ia} \in V(\omega)\}.$$

Obviously, V_a is a closed interval or empty set $\emptyset$. Clearly, it suffices to show that for every $z = re^{ia}$ (r — real, $0 \leqslant a < \pi$) there is

$$\{\omega \mid r \in V_a(\omega)\} \in \mathscr{L}.$$

Put

$$\varphi_a(\omega) = \begin{cases} \sup\{x|x \in V_a(\omega)\} & \text{if } V_a \neq \emptyset, \\ -\infty & \text{if } V_a = \emptyset, \end{cases}$$

and

$$\psi_a(\omega) = \begin{cases} \inf\{x|x \in V_a(\omega)\} & \text{if } V_a \neq \emptyset, \\ +\infty & \text{if } V_a = \emptyset. \end{cases}$$

Then

$$\{\omega|r \in V_a(\omega)\} = \{\omega|\psi_a(\omega) \leqslant r \leqslant \varphi_a(\omega)\}$$

and it remains to prove measurability of functions φ_a and ψ_a.

We shall show the measurability of φ_a. For that purpose let N_n denote the set of all random variables of the form

$$\mu(\omega) = \sum_{i=1}^{m} \lambda_i \xi_{k_i}(\omega),$$

where $m = 1, 2, \ldots, \lambda_i$ — rational, $\lambda_i > 0$, $\sum_{i=1}^{m} \lambda_i = 1$, and $k_i \geqslant n$. (N_n is a dense "rational" part of V_n).

For $0 \leqslant a < \pi$, $n = 1, 2, \ldots$, $k = 1, 2, \ldots$ put

$$N_{n,k}^a = \left\{\mu = r_\mu \exp(ia_\mu)|\mu \in M_n, \quad r_\mu - \text{real}, \ |a_\mu - a| < \frac{1}{k}\right\}$$

and

$$M_{n,k}^a = N_{n,k}^a \cup \{e\}, \qquad \text{where } e(\omega) \equiv -\infty.$$

Measurability of φ_a follows from the equality

$$\varphi_a(\omega) = \inf_{n=1,2\ldots} \inf_{k=1,2\ldots} \sup_{\mu \in M_{n,k}^{a}} r_\mu(\omega),$$

where

$$\mu(\omega) = r_\mu(\omega)\exp(ia_{\mu(\omega)}),$$

$r_{\mu(\omega)}$ — real and $0 \leqslant a_{\mu(\omega)} < \pi$.

The proof of the measurability of ψ_a is analogous.

2. The cores

$$V(\omega) = \operatorname{core} \xi_\nu(\omega)$$

may exhaust all the closed convex subsets of the plane. We shall show somewhat more. Let a probability space be the segment $\Omega = [0, 1]$ with the usual Lebesgue measure. We shall construct a sequence $\{\xi_\nu\}$

(*) $\{\xi_\nu\}$ is convergent almost everywhere on Ω,

(**) for every closed convex subset V of the plane Z there exists a number $\omega \in \Omega$ such that $\operatorname{core}_k \xi_k(\omega) = V$.

Let

$$f_n(\omega) = e^{2\pi i \vartheta_n(\omega)} \quad (n = 1, 2, \ldots, \ 0 \leqslant \omega \leqslant 1)$$

be the orthogonal Steinhaus system ([3], p. 134), i.e., the functions ϑ_n are defined in the following manner: if $\omega = [0, \tau_1 \tau_2, \ldots]_2$ is the infinite dyadic expansion of the number $\omega \in [0, 1]$, then

$$\begin{aligned} \vartheta_1(\omega) &= [0, \tau_1 \tau_3 \tau_6 \ldots]_2, \\ \vartheta_2(\omega) &= [0, \tau_2 \tau_5 \ldots]_2, \\ \vartheta_3(\omega) &= [0, \tau_4 \ldots]_2, \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{3}$$

Let $\{a_k\}$ be a sequence of positive numbers satisfying conditions

$$\text{(i)} \quad \sum_{n=1}^{\infty} a_n = \infty \quad \text{and} \quad \text{(ii)} \quad \sum_{n=1}^{\infty} a_n^2 < \infty. \tag{4}$$

Put

$$\xi_n(\omega) = \sum_{k=1}^{n} a_k f_k(\omega). \tag{5}$$

THEOREM 2. *A sequence of random variables $\{\xi_n\}$ defined by* (5) *possesses both properties* (*) *and* (**).

Proof. By (4) (ii) and a theorem of Steinhaus ([3], p. 137), the series $\sum_{n=1}^{\infty} a_n f_n(\omega)$ converges almost everywhere on $\Omega = [0, 1]$. To show (**) take a closed convex subset V of the plane Z.

By (4) there exists a sequence of "directions" $\{\alpha_k\}$ $(0 \leqslant \alpha_k \leqslant 1)$ such that

$$V = \operatorname*{core}_k \Big\{ \sum_{\nu=1}^{k} a_k e^{2\pi i \alpha_\nu} \Big\}.$$

Put

$$\alpha_k = \vartheta_k(\omega) \quad \text{for } k = 1, 2, \ldots \tag{6}$$

The equalities (6) and (3) determine the number $\omega \in [0, 1]$ such that

$$\operatorname*{core}_k \xi_k(\omega) = V,$$

which completes the proof.

It is not difficult to prove that the sequence (5) is divergent on a residual subset of the interval $[0, 1]$.

3. A random convex set (1) (or stochastic process (2)) describes an asymptotic behavior at infinity of the sequence of random variables $\{\xi_k\}$. Restrictions imposed on the stochastic process (2) are requirements concerning the sequence $\{\xi_k\}$. In general theory of stochastic processes one of the natural requirements imposed on the process is its continuity (in probability or with probability one). We shall prove the following

THEOREM 3. *Process* (2) *is continuous in probability at the point* $z \in Z$ *if and only if*

$$\mathrm{Prob}\{\omega | z \in \mathrm{Fr}\, V(\omega)\} = 0,$$

where Fr A *denotes the boundary of the set* A. *The process* (2) *is continuous in probability if and only if it is continuous with probability one.*

Proof. Let $z \in Z$. Put $\Omega_z = \{\omega | z \in \mathrm{Fr}\, V(\omega)\}$. (The measurability of the set Ω_z and of the set M_a defined below may be proved analogously as in the proof of theorem 1.)

If $\omega \notin \Omega_z$, then either (a) $z \notin V(\omega)$ or (b) $z \in \mathrm{Int}\, V(\omega)$. Let $z_n \to z$. Then, in the case (a) $\eta(z, \omega) = 0$ and $\eta(z_n, \omega) = 0$ for $n > n_0(\omega)$. Similarly, in the case (b), $\eta(z, \omega) = 1$ and $\eta(z_n, \omega) = 1$ for n sufficiently large. If $\mathrm{Prob}(\Omega_z) = 0$, then $\mathrm{Prob}\big(\omega | \eta(z_n, \omega) \to \eta(z, \omega)\big) = 1$, thus the process (2) is continuous with probability one.

Now, let $\mathrm{Prob}(\Omega_z) = \gamma > 0$. For the simplicity assume $z = 0$ and for $0 \leqslant a < 2\pi$ put

$$R_a = \{re^{ia} | r \geqslant 0\},$$

$$M_a = \{\omega \in \Omega_z | V(\omega) \cap R_a \neq \emptyset\}.$$

We shall show that there exists an a such that $\mathrm{Prob}(M_a) < \gamma$. For if to the contrary, for almost every $\omega \in \Omega_z$ there would exist positive numbers $r_k(\omega)$ $(k = 1, 2, 3, 4)$ such that $r_k(\omega) e^{ik\pi/2} \in V(\omega)$ for $k = 1, 2, 3, 4$, then it would follow by the convexity of V that $z = 0 \in \mathrm{Int}\, V$, which is impossible. Thus $\mathrm{Prob}(M_a) < \gamma$ for some a. For such an a we have

$$\mathrm{Prob}\{\omega \in \Omega_z | (1/n) e^{ia} \notin V(\omega)\} \geqslant \gamma - \mathrm{Prob}(M_a) > 0,$$

whence

$$\mathrm{Prob}\{\omega \,\big|\, |\eta\big((1/n) e^{ia}, \omega\big) - \eta(0, \omega)| \geqslant 1\} \geqslant \gamma - \mathrm{Prob}(M_a).$$

Thus the process (2) is not continuous in probability at the point $z = 0$, which ends the proof.

Let λ stand for the Lebesgue measure on the plane Z. If the process (2) is λ-almost everywhere continuous in probability, then, considering if necessary a process stochastically equivalent to the process (2), we may assume that the process (2) is measurable (see [1] or [4]). Then the formula

$$\zeta(\omega) = \int_{V(\omega)} \eta(z, \omega) \lambda(dz) = \lambda\big(V(\omega)\big) \quad \text{for } \omega \in \Omega$$

defines a random variable ζ. The expected value $E\zeta$ may be treated as the (two-dimensional) coefficient of divergence of the sequence $\{\xi_\nu\}$. If for a random variable ξ we put

$$\eta(z, \omega) = \begin{cases} 1 & \text{for } \xi(\omega) = z, \\ 0 & \text{for } \xi(\omega) \neq z \end{cases} \tag{7}$$

(in the case when $\xi_\nu \equiv \xi$), then the process (7) is continuous everywhere except for at most denumerable set of values $\{z_j\}$ for which $\mathrm{Prob}(\omega | \xi(\omega) = z_j) > 0$.

4. The problematics presented here may be easily extended to the case of the processes $\{\xi_t\}$ with values in R^n and with continuous time. Let $(\xi_t, t \geqslant 0)$ be such a process. Put

$$V_\xi(\omega) = \bigcap_{s \geqslant 0} \overline{\mathrm{conv}}\{\xi(t, \omega) | t \geqslant s\},$$

where $\overline{\mathrm{conv}}(A)$ denotes the smallest closed convex set containing set A. V is called a core at infinity of the process ξ. For instance, the theorem 1 from section 1 may be formulated in this context as follows.

THEOREM 4. *If a process $(\xi_t, t \geqslant 0)$ is separable* ([4], *p.* 504), *then for every $z \in Z$ the formula*

$$\eta_z(\omega) = \begin{cases} 1 & \text{if } z \in V_\xi(\omega), \\ 0 & \text{if } z \notin V_\xi(\omega) \end{cases}$$

defines a random variables.

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