COLLOQUIUM MATHEMATICUM

VOL. LIX 1990 FASC. 1

SOME FACTS ABOUT HOMOGENEITY PROPERTIES

BY

JUDY KENNEDY (AUBURN, ALABAMA)

It has been known for some time that if X is a compact metric space, then $H(X)$, the set of all homeomorphisms from X onto itself, is a complete separable metric topological group which acts on X (see [4]). In recent years, a theorem due to E. G. Effros (see [9]) has come to the attention of topologists and has proven to be a very valuable tool for working with spaces of homeomorphisms and their actions on continua.

In order to use Effros' theorem one needs to have certain "nice" properties on both the subgroup G of $H(X)$ involved and the orbit of X under the action of G involved. Here, we obtain results about the action of $H(X)$, or a subgroup of $H(X)$, on X, doing what we can without Effros' theorem when it is not applicable, and using it when it is applicable along with techniques developed largely by Ungar [22], [23]. It should be noted here also that Gerald Ungar was the first topologist to use and recognize the importance of Effros' theorem.

The author would like to thank the referee for his thorough, careful job and the shortening and improvement of several proofs and results which appear in the paper.

I. Background, definitions, notation. We present here a brief discussion of the ideas involved in this paper. For a more detailed discussion, the reader is referred to [19].

If X is a compact metric space, $H(X)$ will denote the set of all homeomorphisms from X onto itself. The topology on $H(X)$ is the compact-open topology. A metric which is compatible with this topology is the commonly used sup metric. If d is a metric on X compatible with its topology, define ϱ, the sup metric on $H(X)$, as follows: if $f, g \in H(X)$,

$$\varrho(f, g) = \mathrm{lub}\{d(f(x), g(x)) \mid x \in X\}.$$

Also, if $\varepsilon > 0$, $x \in X$, $f \in H(X)$, let

$$D_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\} \quad \text{and} \quad N_\varepsilon(f) = \{g \in H(X) \mid \varrho(f, g) < \varepsilon\}.$$

Note that ϱ, $D_\varepsilon(x)$, $N_\varepsilon(f)$ all depend on the given metric d on X. We will be somewhat careless about our notation when no confusion arises. In particular, when we speak of a compact metric space X, d will be the name of the metric we put on X, whether it is mentioned or not, and if $\varepsilon > 0$, $x \in X, f \in H(X)$, then ϱ, $D_\varepsilon(x)$, $N_\varepsilon(f)$ will have the meanings described here.

Note that, for X compact metric, subgroups of $H(X)$ act also on X. The action of the group G on X is *transitive* if for x, y in X there is some g in G such that $g(x) = y$. If $x \in X$, $A \subseteq G$, let

$$Ax = \{g(x) \mid g \in A\}.$$

For each x in X, Gx is known as the *orbit* of x in X under the action of G. We will use 1 to denote the identity in $H(X)$. Also, $G(x) = Gx$.

In this paper, a *continuum* will be a compact, connected, metric space. We will use N to denote the positive integers. If A is a collection of sets, A^* will denote the union of the members of A.

If $n \in N$,

$$F^n(X) = \{x = (x_1, x_2, \dots, x_n) \in X^n \mid \text{if } i \neq j,\ i, j \in \{1, \dots, n\},\ x_i \neq x_j\}.$$

If G is a subgroup of $H(X)$, we will take the action of G on $F^n(x)$ as follows: for $h \in G$, $x = (x_1, \dots, x_n) \in F^n(X)$, define

$$h(x) = (h(x_1), \dots, h(x_n)).$$

A space X is *strongly n-homogeneous* (*n-homogeneous*) if whenever

$$(x_1, x_2, \dots, x_n),\ (y_1, y_2, \dots, y_n) \in F^n(X)$$

there is some h in $H(X)$ such that

$$h(x_i) = y_i \quad \text{for each } i \leqslant n \ (h\{x_1, \dots, x_n\} = \{y_1, \dots, y_n\}).$$

Ungar [22] has shown that if X is an n-homogeneous continuum, then X is strongly n-homogeneous or X is the circle.

A space X is said to be *nearly n-homogeneous* if whenever $\{x_1, \dots, x_n\}$ is an n-element subset of X and $\{D_1, \dots, D_n\}$ is a collection of n open subsets of X, there are an h in $H(X)$ and an n-element subset $\{y_1, \dots, y_n\}$ of X such that

(1) $y_i \in D_i$ for each $i \leqslant n$;

(2) $h\{x_1, \dots, x_n\} = \{y_1, \dots, y_n\}$.

(If $n = 1$, the terms homogeneous and nearly homogeneous are used.)

The following is a version of the Effros theorem due to Ancel [1]:

Suppose that a separable complete metric topological group G acts transitively on a metric space X. Then the following are equivalent:

(1) G acts micro-transitively on X;

(2) X has a complete metric;

(3) X is a second category in itself.

The action of G on X is *micro-transitive* on X if, for every x in X and every neighborhood u of 1 in G, ux is a neighborhood of x in X. (Note: It is easy to show that this is the same as saying that if u is open in G, ux is open in X.)

A map α to a space X from a simple closed curve S is *essential* if it is not homotopic to a constant, and is *inessential* if it is homotopic to a constant.

II. Somewhat without Effros.

Remarks. Suppose G is a complete subgroup of $H(X)$, where X is a compact metric space. Then Gx may not be complete, but Gx is a Borel subset of X, at least (see [14]). Also, G is complete if and only if G is closed in $H(X)$.

THEOREM 1. *Suppose X is a compact metric space, G is a subgroup of $H(X)$, $x \in X$ such that Gx is uncountable. Then, if $\varepsilon > 0$, then $(G \cap N_\varepsilon(1))(x)$ is uncountable.*

Proof. Since G is second countable, some countable subcollection of

$$\{h(N_\varepsilon(1) \cap G) \mid h \in G\}$$

covers G. List the elements of some such subcollection: $h_1 A$, $h_2 A$, ... (where $A = N_\varepsilon(1) \cap G$). Since Gx is uncountable, there is i such that $h_i A(x)$ is uncountable. But then $h_i^{-1}(h_i A(x)) = A(x)$ is also uncountable.

EXAMPLE 2. There is an example of a compact metric space L such that

(1) L admits a countable dense orbit under the action of $H(L)$;

(2) if x is in the countable dense orbit, there is $\varepsilon > 0$ such that $N_\varepsilon(1)(x) = \{x\}$.

The idea behind this example is quite simple: It is a simple closed curve which has been "pinched" at a countable number of places. Describing the example precisely and proving it has the properties claimed seems to be a rather messy business, however. Thus, we will just describe the example, but omit the proof that it has the properties claimed, since it is not difficult to believe that this example has these properties, nor to construct other examples, and the proof is long and technical.

Construction of the Example. Consider the unit interval $[0, 1]$ and let $Q = \{q_1, q_2, \ldots\}$ denote the rationals in $(0, 1)$. Let $q_1 = d_1$ and let d'_1 denote the first element of Q larger than d_1. Now $[0, 1] - \{d_1, d'_1\}$ consists of exactly 3 mutually exclusive intervals c_{11}, c_{12}, c_{13}. Let d_2 denote the first element of Q not d_1 or d'_1. Let C denote the element of $\{c_{11}, c_{12}, c_{13}\}$ which contains d_2, and let d'_2 denote the first element of Q larger than d_2 also in C. Continue this process inductively: Suppose that for $n \in N$, the pairs

$$\{\{d_1, d'_1\}, \{d_2, d'_2\}, \ldots, \{d_n, d'_n\}\}$$

have already been chosen so that, given any pair $\{d_i, d'_i\}$ from this collection,

$$d_i < d'_i$$

and, given another pair $\{d_j, d'_j\}$ such that $i < j$, either

$$d'_j < d_i, \quad d_i < d_j < d'_j < d'_i$$

or

$$d'_i < d_j.$$

Then choose $\{d_{n+1}, d'_{n+1}\}$ from Q as follows: Let d_{n+1} denote the first element of Q not in

$$Q-\{d_1, d'_1, d_2, d'_2, \ldots, d_n, d'_n\}.$$

Now $[0, 1]-\{d_1, d'_1, \ldots, d_n, d'_n\}$ consists of a finite set of mutually exclusive intervals, of which d_{n+1} is in exactly one. Call it C_{n+1}. Choose d'_{n+1} to be the first element of Q in C_{n+1} which is larger than d_{n+1}. Let

$$E = \{\{d_i, d'_i\} \mid i \in N\},$$

$$F = \{\{x\} \mid x \in (0, 1)-E^*\}, \qquad F' = \{\{x\} \mid x \in (0, 1)\}.$$

Now $F' \cup \{\{0, 1\}\}$ gives a decomposition of $[0, 1]$ into a family of closed nonempty disjoint sets. This decomposition, endowed with the usual quotient topology, is the simple closed curve S. Then $E \cup F \cup \{\{0, 1\}\}$ is a decomposition of S into a family of closed nonempty disjoint sets and this decomposition endowed with the usual quotient topology is upper semicontinuous. Thus the resulting space L is a compact metric space. Suppose that P': $[0, 1] \to S$ is the projection map, and likewise P: $S \to L$ is the projection map. Note that if $\{d_i, d'_i\}$, $\{d_j, d'_j\} \in E$; $i < j$, then the set $\{d_i, d'_i\}$ does not separate d_j and d'_j in S.

Bellamy [5] has pointed out the following to the author and given his permission for it to be included here:

THEOREM 3. *Suppose that the separable complete metric topological group G acts on the compact metric space X and that there is $y \in X$ such that Gy is countable. If $x \in Gy$, then there is some open set o in G such that $1 \in o$ and $o(x) = \{x\}$.*

Proof. If $G_y = \{g \in G \mid g(y) = y\}$, then G_y is a closed subgroup of G and $\{hG_y \mid h \in G\}$ is a partition of G into a countable collection of closed homeomorphic disjoint sets. Since G is complete, some hG_y has interior in G, and thus each hG_y has interior. The theorem follows.

Suppose that X is a compact metric space, G is a subgroup of $H(X)$, and o is an open subset of G such that $o = o^{-1}$ (i.e., $h \in o$ iff $h^{-1} \in o$) and $1 \in o$. Then G_o will denote

$$\{k \in G \mid \text{there is a finite subset } \{k_1, k_2, \ldots, k_n\} \text{ of } o \text{ such that } k = k_n \circ k_{n-1} \circ \ldots \circ k_1\}.$$

Note that G_o is a closed-open subgroup of G.

THEOREM 4. *Suppose X is a compact metric space. For $x \in X$ and G a (complete) subgroup of $H(X)$, let*

$$Cx = \{y \in X \mid \textit{for each open subset } o \textit{ of } G \textit{ such that } o = o^{-1} \textit{ and } 1 \in o,\ y \in G_o(x)\}$$

Then if $x_o \in X$, $C = \{Cx \mid x \in G(x_o)\}$ decomposes $G(x_o)$ into a collection of disjoint

(Borel) sets, and if x, y are in $G(x_o)$, and if $h \in G$ and $h(x) = y$, then

$$h(Cx) = Cy.$$

Further, if $R_{x_o} = \{h \in G \mid h(x_o) \in Cx_o\}$, then R_{x_o} is a closed subgroup of G.

Proof. Suppose that x, y are in $G(x_o)$ such that $Cx \cap Cy \neq \emptyset$. Let

$$O = \{o \text{ open in } G \mid o = o^{-1} \text{ and } 1 \in o\}.$$

There is some z in $Cx \cap Cy$ and if $o \in O$, there are $k, h \in G_o$ such that $k(x) = z$, $h(y) = z$. Then $h^{-1}k(x) = y$ and since $h^{-1}k \in G_o$ also, $y \in Cx$. In fact, $Cy \subseteq Cx$ and $Cx \subseteq Cy$, so that $Cx = Cy$.

If x, y are in $G(x_o)$, there is h in G such that $h(x) = y$. Then $h(Cx) = Ch(x) = Cy$: Now $y \in Ch(x) \cap h(Cx)$. Suppose $w \in h(Cx)$. Then $w = h(w')$ for some $w' \in Cx$, and if $o \in O$, there is some l_o in G_o such that $l_o(x) = w'$. But then

$$hl_oh^{-1}(y) = w.$$

Since $\{hoh^{-1} \mid o \in O\} = O$, $w \in Cy$ and $h(Cx) \subseteq Cy$.

Suppose $z \in Cy$. For each o in O, there is l_o in G_o such that $l_o(y) = z$. Then

$$h^{-1}l_oh(x) = h^{-1}(z) \quad \text{and} \quad h^{-1}l_oh \in G_{h^{-1}oh}.$$

Again, since $\{h^{-1}oh \mid o \in O\} = O$, this means that $h^{-1}(z) \in Cx$ and $z \in h(Cx)$. Thus

$$Cy \subseteq h(Cx) \quad \text{and} \quad Cy = h(Cx) = C(h(x)).$$

Suppose $x \in G(x_o)$. Now, if G is complete, then for each o in O, G_o is complete, and so $G_o(x)$ is Borel. Also,

$$Cx = \bigcap_{o \in O} G_o(x),$$

and if $\{o_1, o_2, \ldots\}$ is a subcollection of O which gives a neighborhood base for 1, then

$$\bigcap_{o \in O} G_o(x) = \bigcap_{i \in N} G_{o_i}(x).$$

Thus Cx is a countable intersection of Borel sets, and so is itself Borel.

Now R_{x_o} is a subgroup of G since

(1) $1 \in R_{x_o}$;

(2) $f, g \in R_{x_o}$ means $f \circ g \in R_{x_o}$ (since $g(x_o) \in Cx_o$, $Cg(x_o) = gCx_o = Cx_o$ and likewise $fg(x_o) \in Cg(x_o) = Cx_o$, so $Cx_o = Cf \circ g(x_o)$);

(3) $h \in R_{x_o}$ means $h^{-1} \in R_{x_o}$.

Suppose $h \in \bar{R}_{x_o}$ and $h(x_o) = z$. We wish to show that $z \in Cx_o$. Let $o \in O$ and note that $h^{-1} \in \bar{R}_{x_o}$. Since $h^{-1}o$ is an open set containing h^{-1}, there is $r^{-1} \in R_{x_o}$ such that $r^{-1} \in h^{-1}o$. Thus $hr^{-1} \in o^{-1} = o$. Since $r \in R_{x_o}$, there are $g_1, \ldots, g_n \in o$ such that

$$r(x_o) = g_1 \circ \ldots \circ g_n(x_o).$$

Therefore, $z = h(x_o) = (hr^{-1})(g_1 \circ \ldots \circ g_n)(x_o) \in G_o(x_o)$.

THEOREM 5. *Suppose H is a separable metric topological group which acts transitively on the topological space $(X, \mathscr{T})$. Then if ϱ is a right invariant metric on H (i.e., $\varrho(h,f) = \varrho(hg, fg)$ for $f, g, h \in H$), ϱ induces a metric d on X. The topology of $(X, |d|)$ is finer than the topology of $(X, \mathscr{T})$ and $(X, |d|)$ is separable. If, in addition, H is a completely metrizable space, then $(X, |d|)$ is a completely metrizable, separable space. If $(X, \mathscr{T})$ is itself a completely metrizable separable space, then $(X, \mathscr{T}) = (X, |d|)$.*

Proof. Define $d: X \times X \to R^+$ as follows:

$$d(x, y) = \operatorname{glb}\{\varepsilon > 0 \mid y \in N_\varepsilon(1)(x)\}.$$

It is routine to check that d is a metric on the set X, that the topology of $(X, |d|)$ is finer than the topology of $(X, \mathscr{T})$, and that $(X, |d|)$ is separable.

Suppose H is complete. Fix $x \in X$. The map

$$T_x: H \to (X, |d|)$$

defined by $T_x(h) = h(x)$ is continuous. It is also open: Suppose $\varepsilon > 0, f \in H$. Consider $N_\varepsilon(f)(x)$. Since ϱ is right invariant,

$$N_\varepsilon(f) = N_\varepsilon(1) \circ f.$$

Let $f(x) = y$. Then $N_\varepsilon(1)(y) = \{z \in X \mid d(z, y) < \varepsilon\}$ is open in $(X, |d|)$, and it follows that T_x is open. By a theorem of Sierpiński [21], $(X, |d|)$ is a completely metrizable space. The last statement follows from Ancel's version of the Effros Theorem [1].

Remark. In the case of a compact metric space X, the metric on X discussed in Theorem 5 has been studied independently by Charatonik and Maćkowiak [9]. They called it the *Effros metric.*

At this point, one might wonder whether the R_{x_0} above is

$$\bigcap_{i \in N} G_{o_i},$$

in general. The answer to that question is no, as we show below, and this answers in the negative a question asked in [19]. The first part of the proof of the following theorem is much like that of Lemma 1 in [16], and, in fact, some phrases are actually quotes.

THEOREM 6. *Suppose M is the Menger universal curve. Suppose further that*

(1) *$o_1, o_2, \ldots$ is a collection of open subsets of $H(M)$ which gives a neighborhood base for 1 such that $o_i = o_i^{-1}$ for each $i \in N$;*

(2) *for each $i \in N$, $H_{o_i} = \{h \in H(M) \mid h = h_1 \circ \ldots \circ h_n\}$ for some finite subcollection $\{h_1, \ldots, h_n\}$ of o_i.*

Then M is a homogeneous continuum such that

$$\bigcap_{i \in N} H_{o_i} = \{1\}$$

while, for each x in M, $Cx = \{y \in M \mid$ *for each o open in* $H(M)$ *such that* $1 \in o$ *and* $o = o^{-1}$, $y = h(x)$ *for some* $h \in H_o\} = M$. (Actually M has very nice homogeneity properties (see [2], [6], [20]).)

Proof. For proofs and discussion of the homogeneity properties of M, see the papers cited above. Now $Cx = M$ for $x \in M$ since $H_o(x) = M$ for each o in $H(M)$ such that $1 \in o$ and $o = o^{-1}$. (Otherwise, $H_o(x)$ is open in M and $\{H_o(y) \mid y \in M\}$ decomposes M into mutually exclusive open sets.)

Suppose then that $f \in H(M), f \neq 1$. Suppose M is defined in the standard manner (see [2]) and, for $i = 1, 2, 3$, π_i denotes the projection of M onto one of its Sierpiński curve faces. There is some open set u in M such that

(1) for some $i \in \{1, 2, 3\}$, $\pi_i(u) \cap \pi_i f(u) = \emptyset$,

(2) each of $\pi_i(u)$ and $\pi_i f(u)$ is "contractible in the complement of the other in the plane containing the appropriate Sierpiński curve face of M".

There is a simple closed curve L in u such that $\pi_i(L)$ is a "boundary" curve B in the Sierpiński curve face such that $\pi_i|L\colon L \to B$ is a homeomorphism of L onto B. Let r denote the "retraction of the Sierpiński curve onto B", "projecting radially in the plane containing it".

Now, $r \circ \pi_i|L = \pi_i|L$, and thus $r \circ \pi_i|L$ is a continuous map of a simple closed curve onto another simple closed curve which is essential. Suppose $\varphi\colon S \to L$ and $\varphi'\colon B \to S$ are homeomorphisms where $S = \{z \in R^2 \mid |z| = 1\}$. Then

$$\varphi' \circ r \circ \pi_i \circ \varphi = \alpha$$

is a homeomorphism from S onto S, and is thus essential. Further,

$$\varphi' \circ r \circ \pi_i \circ f \circ \varphi = \beta$$

is an inessential map of S into S.

For $f, g \in S^s (= \{\alpha\colon S \to S \mid \alpha \text{ is continuous}\})$, define

$$\hat{\varrho}(f, g) = \operatorname{lub}\{d(f(x), g(x)) \mid x \in S\}.$$

Observe that the function $F\colon H(M) \to S^s$ defined by

$$F(h) = \varphi' \circ r \circ \pi_i \circ h \circ \varphi$$

is uniformly continuous. Now, if $g_1, g_2 \in S^s$ are sufficiently close, then g_1 and g_2 are homotopic (see [10], p. 316). Suppose $\hat{\varepsilon} > 0$ such that if $\hat{\varrho}(g_1, g_2) < \hat{\varepsilon}$, then g_1 is homotopic to g_2 and suppose $\varepsilon > 0$ such that if $\varrho(h_1, h_2) < \varepsilon$, then

$$\hat{\varrho}(F(h_1), F(h_2)) < \hat{\varepsilon}.$$

Suppose there are $f_1, f_2, \ldots, f_n \in N_\varepsilon(1)$ such that

$$f_n \circ \ldots \circ f_1 = f.$$

Let $f_0 = 1$. Now, for each j,

$$\varphi' \circ r \circ \pi_i \circ f_j \circ f_{j-1} \circ \ldots \circ f_0 \circ \varphi = \beta_j$$

is a map of S to S. Since $\beta_0 = \alpha$ is essential and $\beta_n = \beta$ is inessential, there is least k such that β_k is inessential. Then β_{k-1} is essential and

$$\hat{\varrho}(\beta_k, \beta_{k-1}) < \hat{\varepsilon},$$

so β_k and β_{k-1} are homotopic, a contradiction.

Remark. The referee has pointed out that the above proof generalizes easily to give the same result for the higher dimensional Menger universal continua M_n^{2n+1}.

THEOREM 7. *Suppose M_s is the Sierpiński plane curve. Again, if the groups $H_{o_1}, H_{o_2}, \ldots$ are defined as before, then M_s is a nearly homogeneous continuum such that*

$$\bigcap_{i=1}^{\infty} H_{o_i} = \{1\};$$

$Cx = H(M_s)(x)$ (Cx defined as before for $H(M_s)$) for x in the dense G_δ-orbit of M_s; and Cx is the boundary curve containing x for x not in the dense G_δ-orbit.

Proof. The first part of this statement can be proved with a simple version of the previous proof.

The second part follows from noting that the dense G_δ-orbit of M_s is connected and that, for each boundary curve B in the dense F_σ-orbit, $B = Cx$ for $x \in B$. One can use Corollary 1 of [14] and Theorem 1.2 of [6] to get the last part.

Remarks. Contrast these results with the following:

(1) Lewis [18] has shown that if P denotes the pseudo-arc, $h \in H(P)$ and $\varepsilon > 0$, then h can be written as a composition of ε-homeomorphisms. Then for $H(P)$

$$\bigcap_{i \in N} H_{o_i} = H(P).$$

He has also shown [17] that $H(P)$ contains no nondegenerate continua.

(2) Anderson [3] has shown that, in the case of the Hilbert cube Q, each h in $H(Q)$ is isotopic to the identity. In fact, $H(Q)$ is connected, locally connected, and infinite dimensional.

(3) Although $H(M)$ and $H(M_s)$ (M is the Menger curve, M_s is the Sierpiński curve) are totally disconnected, they are not 0-dimensional [6].

To sum up what has been done here in this section, for G a complete subgroup of $H(X)$, where X is a compact metric space and $x \in X$, Gx may be classified as follows:

(1) Gx is complete, or equivalently second category in itself, in which case Effros' theorem is applicable;

(2) $Gx = Cx$ (Cx defined as previously in this section), in which case $y \in Gx$ means that if $\varepsilon > 0$, there is a sequence $f_1, \ldots, f_m$ of homeomorphisms in $N_\varepsilon(1) \cap G$ such that $f_m \circ \ldots \circ f_1(x) = y$; or

(3) Gx can be decomposed into a nondegenerate collection $C = \{Cy \mid y \in Gx\}$ of disjoint homeomorphic Borel sets.

Consider the dyadic solenoid Σ. The solenoid is a compact abelian topological group. This space contains a continuous homomorph of the reals R as a dense subgroup ([12], p. 114). Now Σ is actually a closed subgroup of $H(\Sigma)$, and thus there is a continuous homomorphism $\alpha: R \to H(\Sigma)$ and R acts on Σ. [In fact, if G is a separable metric group and there is a continuous homomorphism β from G into $H(X)$, where X is a compact metric space, then G acts on X, this action φ being defined by $\varphi(g, x) = \beta(g)(x)$ for $(g, x) \in G \times X$.] Now R is complete, but $\alpha(R)(x)$ is dense in Σ and not complete for some x. Thus $\alpha(R)(x)$ would be an orbit under the action of R which has the property that

$$\alpha(R)(x) = \bigcap_{i=1}^{\infty} \alpha(R_{o_i})(x),$$

where $o_i = (-1/i, 1/i)$ for each i, and

$$R_{o_i} = \{x \in R \mid x = \sum_{i=1}^{n} x_i \text{ for some } \{x_1, \ldots, x_n\} \subseteq o_i\} = R.$$

Now, although $\alpha: R \to H(\Sigma)$ is a continuous homomorphism and, in fact, an isomorphism, $\alpha(R)$ is not homeomorphic to R in $H(\Sigma)$. Also, it is a fact that *topologically* $H(\Sigma)$ does contain dense copies of R (see [13]). But does it contain a copy of R which is both algebraically *and* topologically R? **(P 1381)** Is there G, a complete subgroup of $H(X)$ for some homogeneous continuum X, such that for some x in X, Gx is of type (2) in the preceding classification, but not of type (1)? **(P 1382)**

Why would anyone be interested in orbits of compact metric spaces to which Effros cannot be applied? Other than the obvious fact that there are many interesting continua which are not homogeneous or which admit some orbits which are not complete, there is this consideration: Often when one is trying to decide what stronger homogeneity properties a given homogeneous continuum X has, one looks at how $H(X)$ acts on spaces *other* than X, e.g., how $H(X)$ acts on $F^n(X)$ for $n \in N$ (see [23]).

III. Characterizations of n-homogeneity and near n-homogeneity. The following theorem uses the Effros result very heavily. It also uses techniques developed largely by Ungar [22], [23].

THEOREM 8. *Suppose X is a continuum and $n \in N$. Then the following are equivalent:*

(1) *X is n-homogeneous.*

(2) *If A' is a nowhere dense subset of $F^n(X)$ and $x \in F^n(X)$, $\varepsilon > 0$, there is some $h \in N_\varepsilon(1) \subseteq H(X)$ such that $h(x) \notin A'$.*

(3) *If A is a first category subset of $F^n(X)$ and D is a countable subset of $F^n(X)$, there is a homeomorphism h in $H(X)$ such that $h(D) \cap A = \emptyset$.*

(4) *If A is a first category subset of $F^n(X)$ and $x \in F^n(X)$, there is a homeomorphism h in $H(X)$ such that $h(x) \notin A$.*

The proof of this theorem will follow from the proofs of the next theorems.

THEOREM 9. *Suppose X is an n-homogeneous continuum, A is a first category subset of $F^n(X)$ and D is a countable subset of $F^n(X)$. Then there is some h in $H(X)$ such that $h(D) \cap A = \emptyset$.*

Proof. Since X is homogeneous, a theorem of Burgess [8] gives the result that either X is a simple closed curve or no finite set of points separates X. If no set of $n-1$ points separates X, Lemma 3.9 and Theorem 3.8 of Ungar [22] give the result that $F^n(X)$ is connected and X is strongly n-homogeneous. Then X is either a simple closed curve or X is strongly n-homogeneous.

First suppose X is strongly n-homogeneous and suppose $x \in F^n(X)$. Then

$$T_x\colon H(X) \to F^n(X)$$

defined by

$$T_x(h) = h(x) \ \big(= \big(h(x_1), \ldots, h(x_n)\big), \text{ where } x = (x_1, \ldots, x_n)\big)$$

is a continuous, onto, open mapping by Effros' theorem. Now A is first category in $F^n(X)$, so there is some F_σ-set A' in $F^n(X)$ such that A' is first category and A' contains A. Let

$$E = F^n(X) - A'.$$

Since E is G_δ in $F^n(X)$, $T_x^{-1}(E)$ is dense G_δ in $H(X)$. (Recall that T_x is continuous and open.)

Consider $\bigcap \{T_x^{-1}(E) \mid x \in D\} = K$. Then, by the Baire Category Theorem, K is dense G_δ in $H(X)$. Choose h from K. If $d \in D$, then $h(d) \in E$ and $h(d) \notin A$. Thus $h(D) \cap A = \emptyset$.

Now suppose X is a simple closed curve and suppose $x \in F^n(X)$. Then

$$T_x\colon H(X) \to F^n(X)$$

defined by

$$T_x(h) = h(x) \ \big(= \big(h(x_1), \ldots, h(x_n)\big), \text{ where } x = (x_1, \ldots, x_n)\big)$$

is a continuous, open (but not onto) mapping by Effros' theorem. Now A is first category in $F^n(X)$, so there is some F_σ-set A' in $F^n(X)$ such that A' is first category and A' contains A. Let

$$E = F^n(X) - A'.$$

Since E is dense G_δ in $F^n(X)$, $T_x^{-1}(E)$ is dense G_δ in $H(X)$. (Recall that T_x is continuous and open, and $E \cap T_x(H(X))$ must be dense G_δ in $\overline{T_x(H(X))}$.)

Consider $\bigcap \{T_x^{-1}(E) \mid x \in D\} = K$. By the Baire Category Theorem, K is dense G_δ in $H(X)$. Choose h from K. If $d \in D$, then $h(d) \in E$ and $h(d) \notin A$. Thus $h(D) \cap A = \emptyset$.

THEOREM 10. *Suppose that X is a continuum. Suppose that if A is a first* *category subset of $F^n(X)$ and $x \in F^n(X)$, then there is some h in $H(X)$ such that $h(x) \notin A$. Then X is strongly n-homogeneous or X is a simple closed curve.*

Proof. Suppose that $x \in F^n(X)$. Denote $H(X)(x)$ simply by Hx. Now, Hx cannot be a first category set in $F^n(X)$, for if so, then there is some h in $H(X)$ such that $h(x) \notin Hx$, and this cannot happen. Then Hx is a second category subset of $F^n(X)$, and is thus second category in itself. Effros' theorem then implies that each orbit is a G_δ-set in $F^n(X)$.

But, since $x \in F^n(X)$ implies that Hx is second category in $F^n(X)$, we also know that $(\overline{Hx})^\circ \neq \emptyset$. Since $h \in H(X)$ implies $h((\overline{Hx})^\circ) = (\overline{Hx})^\circ$, we have $Hx \subseteq (\overline{Hx})^\circ$: Suppose $z \in (\overline{Hx})^\circ \cap Hx$ and $t \in Hx$. There is some $\hat{h}$ in $H(X)$ such that $\hat{h}(z) = t$, and $t \in (\overline{Hx})^\circ$. (If A is a set, A° denotes the interior of A.)

Now, if x and y are points of $F^n(X)$ such that $Hx \cap Hy = \emptyset$, then also

$$(\overline{Hx})^\circ \cap (\overline{Hy})^\circ = \emptyset.$$

Otherwise, $(\overline{Hx})^\circ \cap (\overline{Hy})^\circ$ is a nonempty open set, and

$$Hx \cap \left((\overline{Hx})^\circ \cap (\overline{Hy})^\circ\right) \quad \text{and} \quad Hy \cap \left((\overline{Hx})^\circ \cap (\overline{Hy})^\circ\right)$$

are mutually exclusive dense G_δ-sets in $(\overline{Hx})^\circ \cap (\overline{Hy})^\circ$. This is impossible. Then

$$Hx = \overline{Hx} = (\overline{Hx})^\circ.$$

Indeed, suppose $z \in \overline{Hx}$. Then

$$z \in Hz \subseteq (\overline{Hz})^\circ \quad \text{and} \quad (\overline{Hz})^\circ \cap (\overline{Hx})^\circ \neq \emptyset,$$

and so $Hz = Hx$.

If $n = 1$, we can conclude that X is homogeneous, since X is a continuum and does not consist of a nondegenerate collection of disjoint open-closed sets. Suppose $n > 1$. We can still conclude that if $x \in X$, then $H(X)(x)$ is open. (H is acting on X now, not $F^n(X)$. Since $z = (z_1, \ldots, z_n)$ is in $F^n(X)$ implies that $H(z_1, \ldots, z_n)$ is open in $F^n(X)$, there is some open set $u = u_1 \times u_2 \times \ldots \times u_n$ such that $z_i \in u_i$ for each $i \leqslant n$ and $u \subseteq H(z_1, \ldots, z_n)$. Then $u_i \subseteq H(X)(z_i)$ for each $i \leqslant n$, and thus $H(X)(z_i)$ is open in X.) But then X is homogeneous, for it is a continuum.

Then we know from a theorem of Burgess [8] that either X is a simple closed curve or no set of $n-1$ points separates X. If no set of $n-1$ points separates X, Lemma 3.9 and Theorem 3.8 of Ungar [22] imply that $F^n(X)$ is connected and X is strongly n-homogeneous. Then X is either a simple closed curve or X is strongly n-homogeneous.

COROLLARY 11. *Suppose that the continuum X has the property that if A' is a nowhere dense subset of $F^n(X)$ and $x \in F^n(X)$, $\varepsilon > 0$, then there is some $h \in N_\varepsilon(1) \subseteq H(X)$ such that $h(x) \notin A'$. Then X is n-homogeneous.*

Proof. This is just a special case of the situation in Theorem 10. If A' is a nowhere dense subset of $F^n(X)$, then A' is a first category subset of $F^n(X)$. Also, if there is some $h \in N_\varepsilon(1)$ (for some $\varepsilon > 0$) such that $h(x) \notin A'$, then there is $h \in H(X)$ such that $h(x) \notin A'$. Then X is strongly n-homogeneous or X is a simple closed curve by Theorem 10. Since simple closed curves are n-homogeneous, the corollary follows.

To finish off Theorem 8. We have now $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$ and $(2) \Rightarrow (1)$. Since if X is n-homogeneous, X is either a simple closed curve or X is strongly n-homogeneous, and in either case $x \in F^n(X)$ defined by

$$T_x(h) = (h(x_1), \ldots, h(x_n)), \quad \text{where } x = (x_1, \ldots, x_n),$$

is open, we have $(1) \Rightarrow (2)$.

We will say that the space X is *nearly strongly n-homogeneous* if whenever $\{x_1, \ldots, x_n\}$ is an n-element subset of X and $\{v_1, \ldots, v_n\}$ is a set of n open sets of X, there is some $h \in H(X)$ such that $h(x_i) \in v_i$ for each $i \in N$.

THEOREM 12. *Suppose that X is a continuum, $n \in N$, and $F^n(X)$ is connected. Then the following are equivalent*:

(1) *X is nearly n-homogeneous.*

(2) *If A is a nowhere dense subset of $F^n(X)$ and $x \in F^n(X)$, there is $h \in H(X)$ such that $h(x) \notin A$.*

(3) *X is nearly strongly n-homogeneous.*

Proof. $(1) \Rightarrow (2)$. Suppose

$$P = \{\varphi \mid \varphi \text{ is a 1-1 function from } \{1, \ldots, n\} \text{onto} \{1, \ldots, n\}\}.$$

Then, for each $\varphi \in P$, let

$$A_\varphi = \{z = (z_1, \ldots, z_n) \in F^n(X) \mid z_i = x_{\varphi(i)}, \text{ where } x = (x_1, \ldots, x_n) \in A\}.$$

Then A_φ is homeomorphic to A for $\varphi \in P$, and since P is finite, the set

$$\hat{A} = \bigcup_{\varphi \in P} A_\varphi$$

is nowhere dense in $F^n(X)$. There is some basic open set $u = u_1 \times \ldots \times u_n$ in $F^n(X)$ such that $u \cap \hat{A} = \emptyset$. Note that

$$u_{\varphi(1)} \times \ldots \times u_{\varphi(n)} = u_\varphi$$

has the property that $u_\varphi \cap \hat{A} = \emptyset$ also for $\varphi \in P$.

Suppose $x \in F^n(X)$, $x = (x_1, \ldots, x_n)$. There is some h in $H(X)$ and $z = (z_1, \ldots, z_n) \in u$ such that

$$h(\{x_1, \ldots, x_n\}) = \{z_1, \ldots, z_n\}.$$

Then $h(x_i) = z_{\varphi(i)}$ for some $\varphi \in P$ and

$$h(x) = (h(x_1), \ldots, h(x_n)) \in u_\varphi.$$

Thus $h(x) \notin A$.

(2) $\Rightarrow$ (3). If $x \in F^n(X)$, then $H(X)(x)$ is not nowhere dense in $F^n(X)$, so there is some open subset u of $F^n(X)$ such that $H(X)(x)$ is dense in u. Let

$$U_x = \{u \text{ open in } F^n(X) \mid H(X)(x) \text{ is dense in } u\}^*.$$

Then $H(X)(x) \subseteq U_x$, and for $z \in H(X)(x)$, $t \in H(X)(x) \cap U_x$ there is some h in $H(X)$ such that $h(t) = z$. Then there is some open $o \subseteq U_x$ such that $t \in o$. Thus $z \in h(o)$ and $H(X)(x)$ is dense in $h(o)$, so $h(o) \subseteq U_x$.

Suppose that $U_x \cap U_y \neq \emptyset$. Suppose

$$w \in H(X)(x) \quad \text{and} \quad w' \in H(X)(x) \cap U_x \cap U_y.$$

Now $w' \in \overline{H(X)(y)}$, and since there is some k in $H(X)$ such that $k(w') = w$, we have $w \in \overline{H(X)(y)}$ and $w \in U_y$. Then $H(X)(x) \subseteq U_y$. Likewise $H(X)(y) \subseteq U_x$. Thus $\bar{U}_x = \bar{U}_y$, and it follows (if one notes that $H(X)(w) \cap \bar{U}_x \neq \emptyset$ implies $U_w \cap U_x \neq \emptyset$ and that $\bar{U}_w = \bar{U}_x$) that $U_x = U_y$. Then $F^n(X)$ can be written as a union of mutually exclusive open sets. But $F^n(X)$ is connected, and this cannot happen unless $U_x = F^n(X)$. It now follows that X is nearly strongly n-homogeneous.

(3) $\Rightarrow$ (1) is obvious.

Remarks. The preceding theorem is an analog to Ungar's result that if $F^n(X)$ is connected, and X is a compact metric space, then X is n-homogeneous iff X is strongly n-homogeneous.

Burgess [7] asked in 1955 whether for $n \geqslant 1$ every n-homogeneous metric continuum is $(n+1)$-homogeneous. The answer to this question is still not known, although some partial results have been obtained:

(1) ([23]) If X is a connected compact metric space, then X is n-homogeneous for all n iff X is countable dense homogeneous.

(2) ([20]) If X is a 2-homogeneous continuum and X admits a stable homeomorphism other than the identity, then X is representable.

(A separable space X is *countable dense homogeneous* if whenever A and B are countable dense subsets of X, there is a homeomorphism h in $H(X)$ such that $h(A) = B$. A *stable homeomorphism* h of the space X onto itself is one which has the property that it is a composition of homeomorphisms of $H(X)$ so that each homeomorphism in the composition is the identity on some non-empty open set. A space X is *representable* means that if $x \in X$, and u is open in X such that $x \in u$, then there is an open set v in X such that (1) $x \in v \subseteq u$; and (2) if $y \in v$, there is h in $H(X)$ such that $h(x) = y$ and $h(z) = z$ for $z \notin v$.)

REFERENCES

[1] F. D. Ancel, *An alternative proof and applications of a theorem of E. G. Effros*, Michigan Math. J. 34 (1987), pp. 39–55.

[2] R. D. Anderson, *A characterization of the universal curve and a proof of its homogeneity*, Ann. of Math. (2) 67 (1958), pp. 313–324.

[3] — *On topological infinite deficiency*, Michigan Math. J. 14 (1967), pp. 365–383.
[4] R. C. Arens, *Topologies for homeomorphism groups*, Amer. J. Math. 58 (1946), pp. 593–610.
[5] D. P. Bellamy, personal communication.
[6] B. L. Brechner, *On the dimensions of certain spaces of homeomorphisms*, Trans. Amer. Math. Soc. 121 (1966), pp. 516–548.
[7] C. E. Burgess, *Homogeneous Continua*, pp. 75–78 in: *Summary of Lectures and Seminars, Summer Institute on Set Theoretic Topology*, University of Wisconsin, 1955.
[8] — *Continua and various types of homogeneity*, Trans. Amer. Math. Soc. 88 (1958), pp. 366–374.
[9] J. J. Charatonik and T. Maćkowiak, *Around Effros' Theorem*, ibidem 298 (1986), pp. 579–802.
[10] J. Dugundji, *Topology*, Allyn and Bacon, Boston 1966.
[11] E. G. Effros, *Transformation groups and C^*-algebras*, Ann. of Math. (2) 81 (1965), pp. 38–55.
[12] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. I*, 2nd edition, Grundlehren Math. Wiss., Band 115, Springer-Verlag, Berlin 1979.
[13] J. Keesling, *The group of homeomorphisms of a solenoid*, Trans. Amer. Math. Soc. 172 (1972), pp. 119–131.
[14] Judy Kennedy, *The application of Ancel's version of Effros' Theorem to non-homogeneous spaces*, Topology Proc. 7 (1982), pp. 245–266.
[15] K. Kuratowski, *Topology. II*, Academic Press, New York 1968.
[16] I. W. Lewis, *Homeomorphism groups and homogeneous continua*, Topology Proc. 6 (1981), pp. 335–344.
[17] — *Pseudo-arcs and connectedness in homeomorphism groups*, Proc. Amer. Math. Soc. 87 (1983), pp. 745–748.
[18] — *Most maps of the pseudo-arcs are homeomorphisms*, ibidem 91 (1984), pp. 147–154.
[19] Judy Kennedy Phelps, *Homogeneity and groups of homeomorphisms*, Topology Proc. 6 (1981), pp. 371–404.
[20] — *A condition under which 2-homogeneity and representability are the same in continua*, Fund. Math. 121 (1984), pp. 89–98.
[21] W. Sierpiński, *Sur une propriété des ensembles G_δ*, ibidem 16 (1930), pp. 173–180.
[22] G. S. Ungar, *On all kinds of homogeneous spaces*, Trans. Amer. Math. Soc. 121 (1975), pp. 393–400.
[23] — *Countable dense homogeneity and n-homogeneity*, Fund. Math. 99 (1978), pp. 155–160.

FAT MATHEMATICS
PARKER HALL
AUBURN UNIVERSITY, AL 36849

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF DELAWARE
NEWARK, DE 19716

Reçu par la Rédaction le 14.12.1987;
en version modifiée le 6.10.1988