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REPRESENTATION OF OPERATORS ON L^1-SPACES BY NONDIRECT PRODUCT MEASURES

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The idea of representing L^1-operators by measurable kernels goes back to Kantorovič and Vulikh [5]. The 1-1 correspondence between L^1-operators and certain measures on the product space was recognized by Grothendieck ([2], p. 62). Similar representation theorems were proved and applied in [1] and [3]. A related theorem is obtained in [4] (Theorem 3.1). In all these results strong topological assumptions (such as compactness or local compactness) are imposed on the underlying measure spaces.

The aim of this note is to clarify the role of compactness by proving a purely measure theoretic version of the representation theorem without any explicit topological assumptions. The notion of perfect measure turns out to be decisive in this context.

Let (X_1, Σ_1, m_1) and (X_2, Σ_2, m_2) be two probability spaces. Given a set function μ defined on $\Sigma_1 \times \Sigma_2$ rectangles we define the marginals of μ by

$$\mu^1(A) = \mu(A \times X_2) \ (A \in \Sigma_1), \qquad \mu^2(B) = \mu(X_1 \times B) \ (B \in \Sigma_2).$$

If μ is positive, (finitely) additive, and satisfies $\mu^i = m_i$, then we say that μ is a *(nondirect) product* of m_1, m_2.

Let B denote the linear space consisting of all linear combinations of the form $\sum a_i f_i \otimes h_i$, where f_i and h_i are characteristic functions of sets in Σ_1 and Σ_2, respectively. B endowed with the sup norm becomes a normed vector lattice. The Banach lattice dual B' is the AL-space of all bounded additive measures (with total variation norm) defined on the algebra generated by $\Sigma_1 \times \Sigma_2$ rectangles. Let J be the lattice ideal in B consisting of all functions which vanish $m_1 \times m_2$ almost everywhere. Clearly, $(B/J)'$ is a sublattice of B' and $\mu \in (B/J)'$ iff μ vanishes on all rectangles of $m_1 \times m_2$ measure zero. Now $\mu(f \otimes h)$ is well defined if f and h are (classes of) simple functions on (X_1, Σ_1, m_1) and (X_2, Σ_2, m_2), respectively.

In $(B/J)'$ we distinguish a subspace $M(m_1, m_2)$ of all $\mu \in B'$ such that the $|\mu|^i$ are countably additive, absolutely continuous with respect to m_i, and $d|\mu|^1/dm_1 \in L^\infty(m_1)$. In particular, all products of m_1 and m_2 are in $M(m_1, m_2)$. Clearly, $M(m_1, m_2)$ is a lattice ideal in $(B/J)'$ and in B'.

The following proposition is a modified version of Theorem 1 in [3]. The proof is essentially the same with obvious simplification of the second part of the argument.

PROPOSITION 1. *The formula* $(T_\mu f, h) = \mu(f\otimes h)$, *where* f *and* h *are simple functions, establishes a lattice isomorphism between* $M(m_1, m_2)$ *and the Banach lattice* $\mathscr{L}(L^1(m_1), L^1(m_2))$ *of all continuous operators from* $L^1(m_1)$ *into* $L^1(m_2)$. *Moreover,* $\|T_\mu\| = \|d|\mu|^1/dm_1\|_\infty$.

If $T_\mu = T$, we shall say that T is *represented* by μ. It is clear that for every $\mu \in M(m_1, m_2)$ the form $\mu(f\otimes h)$ extends uniquely to a continuous bilinear form on $L^1(m_1)\times L^\infty(m_2)$. It should also be remarked that $T'_\mu 1 = d\mu^1/dm_1$ and $T_\mu 1 = d\mu^2/dm_2$.

LEMMA. *Let* m'_i $(i = 1, 2)$ *be a finite positive measure on* Σ_i *absolutely continuous with respect to* m_i. *If every product of* m_1 *and* m_2 *is* σ*-additive, then the same is true for* m'_1 *and* m'_2.

Proof. Let μ be a product of m'_1 and m'_2 and put $f_i = dm'_i/dm_i$. Then $\mu \in (B/J)'_+$ and μ is the (total variation norm) limit of its restrictions to $\{(x, y): f_1(x) \leqslant n, f_2(y) \leqslant n\}$. Since σ-additive measures form a closed (= norm complete) subspace of B', we may assume that, for some n, $f_i \leqslant n$ everywhere.

Let $g_1 = n-f_1$. We define $d\mu_1 = g_1 dm_1\times dm_2$ and $\mu_2 = (\mu+\mu_1)/n$. Clearly, $d\mu_2^2/dm_2 \leqslant 2$. Now let $g_2 = 2-d\mu_2^2/dm_2$. We put $d\mu_3 = dm_1\times g_2 dm_2$ and $\mu_4 = (\mu_2+\mu_3)/2$. Obviously, $d\mu_4^2/dm_2 = 1$ and $d\mu_4^1/dm_1 = (1+\int g_2 dm_2)/2 = \text{const} = 1$ because μ_4 is a probability measure. By assumption, μ_4 is σ-additive. We conclude that μ is σ-additive as a linear combination of the σ-additive measures μ_i.

We have the following consequence of our lemma and Proposition 1:

THEOREM. *Let* (X_i, Σ_i, m_i), $i = 1, 2$, *be probability spaces. Then the following conditions are equivalent*:

(i) *Every product of* m_1 *and* m_2 *is* σ*-additive.*

(ii) *Every additive measure* $\mu \in B'$ *such that the marginals* $|\mu|^i$ *are* σ*-additive and absolutely continuous with respect to* m_i *is* σ*-additive.*

(iii) *Every operator* $T \in \mathscr{L}(L^1(m_1), L^1(m_2))$ *is represented by a* σ*-additive finite measure* μ *on the product* σ*-algebra* $\Sigma_1\times\Sigma_2$.

In [3], (iii) was obtained for Borel probabilities on the unit interval and, more generally, for finite Radon measures on compact Hausdorff spaces. In both cases the measure theoretic compactness (= Marczewski compactness) of m_1 or m_2 allowed us to apply the Marczewski - Ryll-Nardzewski theorem on nondirect products [7] (this idea was earlier used by Brown in [1]). We recall that a finite measure m on (X, Σ) is called *compact* [6] if there exists a family Σ_0 of subsets of X satisfying

(a) for every sequence $A_n \in \Sigma_0$ with $\bigcap A_n = \emptyset$ there exists k such that

$$\bigcap_{n\leqslant k} A_n = \emptyset;$$

(b) for every $A \in \Sigma$ and every $\varepsilon > 0$ there exist $A_0 \in \Sigma_0$ and $B \in \Sigma$ such that

$$B \subset A_0 \subset A \quad \text{and} \quad m(A\setminus B) < \varepsilon.$$

It is well known that every probability space can be extended to a compact space. More precisely, given (X, Σ, m), one can construct an extension $(\bar{X}, \bar{\Sigma}, \bar{m})$ (i.e. $\bar{X} \supset X$, $\bar{\Sigma}|X = \Sigma$, $\bar{m}(\bar{X}) = 1$, and $\bar{m}^*|\Sigma = m$) such that $\bar{m}$ is a compact measure. This can be done, e.g., by Stone compactification.

A property of measures weaker than compactness is perfectness. A finite measure is called *perfect* if all its restrictions to countably generated sub-σ-algebras are compact (see [9] for other equivalent definitions). It was proved by Ryll-Nardzewski [9] that products of perfect measures are always σ-additive. Pachl [8] has shown that perfectness is essentially the weakest possible assumption that guarantees (i). More specifically, if one of the measures, say m_1, is fixed, then (i) holds for an arbitrary probability measure m_2 iff m_1 is perfect. On the other hand, no individual characterization of measures m_1, m_2 satisfying (i) seems to be known even in the case of $m_1 = m_2$. In fact, there exist examples of nonperfect measures $m_1 = m_2$ satisfying (i) (see [8]) ([1]).

PROPOSITION 2. *Let (X_i, Σ_i, m_i), $i = 1, 2$, be probability spaces and let $(\bar{X}_i, \bar{\Sigma}_i, \bar{m}_i)$ be their extensions with at least one of the measures $\bar{m}_1$, $\bar{m}_2$ perfect. Then* (i)-(iii) *are equivalent to*

(iv) $\bar{\mu}^*(X_1 \times X_2) = 1$ *for every product $\bar{\mu}$ of $\bar{m}_1$ and $\bar{m}_2$.*

Proof. (i)$\Rightarrow$(iv). Let $\bar{\mu}$ be a product of $\bar{m}_1$ and $\bar{m}_2$. Then $\bar{\mu}$ is σ-additive by perfectness. We define a set function μ on $\Sigma_1 \times \Sigma_2$ rectangles by

$$\mu\big((\bar{A}_1 \times \bar{A}_2) \cap (X_1 \times X_2)\big) = \bar{\mu}(\bar{A}_1 \times \bar{A}_2), \tag{$*$}$$

where $\bar{A}_i \in \bar{\Sigma}_i$. The definition is correct because if

$$(\bar{A}_1 \times \bar{A}_2) \cap (X_1 \times X_2) = (\bar{B}_1 \times \bar{B}_2) \cap (X_1 \times X_2),$$

([1]) The construction in [8], p. 337, seems to be incorrect. It can be correct if we assume that sets of cardinality less than $2^{\aleph_0}$ have Lebesgue measure zero. The family of all Borel sets whose projections are uncountable should then be replaced by the family of all Borel sets D such that $\lambda(D) \neq 0$ for some nondirect product measure λ. See also J. K. Pachl, *Correction to the paper "Two classes of measures"*, Colloquium Mathematicum 45 (1981), p. 331-333.

then

$$\bar{\mu}\big((\bar{A}_1 \times \bar{A}_2) \dot{-} (\bar{B}_1 \times \bar{B}_2)\big) \leqslant \bar{\mu}\big((\bar{A}_1 \dot{-} \bar{B}_1) \times \bar{X}_2\big) + \bar{\mu}\big(\bar{X}_1 \times (\bar{A}_2 \dot{-} \bar{B}_2)\big)$$
$$= \bar{m}_1(\bar{A}_1 \dot{-} \bar{B}_1) + \bar{m}_2(\bar{A}_2 \dot{-} \bar{B}_2) = 0$$

in view of $(\bar{m}_i)_*(\bar{X}_i \setminus X_i) = 0$. Clearly, $\mu(X_1 \times X_2) = 1$.

By an analogous argument, μ is finitely additive. Since $\mu^i(\bar{A}_i \cap X_i) = \bar{m}_i(\bar{A}_i) = m_i(\bar{A}_i \cap X_i)$, from (i) we infer that μ is σ-additive. Therefore $\mu(\bar{C} \cap (X_1 \times X_2)) = \bar{\mu}(\bar{C})$ for every $\bar{C} \in \bar{\Sigma}_1 \times \bar{\Sigma}_2$, whence $\bar{\mu}^*(X_1 \times X_2) = 1$.

(iv) $\Rightarrow$ (i). Let μ be a product of m_1 and m_2. Then $(*)$ determines a finitely additive measure $\bar{\mu}$ on $\bar{\Sigma}_1 \times \bar{\Sigma}_2$ rectangles. Clearly, $\bar{\mu}^i = \bar{m}_i$, so $\bar{\mu}$ is σ-additive by perfectness. Since $\bar{\mu}^*(X_1 \times X_2) = 1$, the measure $\nu = \bar{\mu}^* \mid (\Sigma_1 \times \Sigma_2)$ is also σ-additive. By $(*)$, ν coincides with μ on $\Sigma_1 \times \Sigma_2$ rectangles, so μ is σ-additive.

It should be noted that (iv) cannot be weakened to the requirement that $(\bar{m}_1 \times \bar{m}_2)^*(X_1 \times X_2) = 1$, since the latter is simply equivalent to $m_i^*(X_i) = 1$ and is always satisfied by assumption.

If m_i' are σ-finite measures on Σ_i, then by the Lemma, (i)-(iv) hold either for all or for none equivalent finite measures m_i. The following proposition shows that the finiteness assumption can be relaxed in the representation theorem.

PROPOSITION 3. *Let (X_i, Σ_i, m_i'), $i = 1, 2$, be σ-finite measure spaces and let m_i be finite equivalent measures. Then* (iii) *is equivalent to*

(iii′) *Every positive operator in $\mathscr{L}\big(L^1(m_1'), L^1(m_2')\big)$ is represented by a positive σ-additive σ-finite measure on $\Sigma_1 \times \Sigma_2$.*

Proof. The mapping $f \to f' = (dm_i/dm_i')f$ is a Banach lattice isomorphism from $L^1(m_i)$ onto $L^1(m_i')$. The formula $T_1 f' = (Tf)'$ defines an isometric isomorphism between the corresponding Banach spaces of operators (the diagram commutes). If (iii) holds and μ represents $T \geqslant 0$, we let $d\mu_1 = \big((dm_1'/dm_1) \otimes 1\big)d\mu$. It is easy to see that the measure μ_1 is positive, σ-finite, and represents T_1 in the sense that

$$\int (T_1 f') h \, dm_2' = \int f' \otimes h \, d\mu_1 \qquad (f' \in L^1(m_1'),\ h \in L^\infty(m_2')),$$

and $\|T_1\| = \|d\mu_1/dm_1\|_\infty$. The converse implication is proved similarly.

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