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EQUATIONALLY COMPACT SEMILATTICES

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1. The concept of equationally compact universal algebras was introduced by Mycielski [2]. In this note we will determine equationally compact semilattices (for the basic concepts see section 2). The main result is the following theorem (1):

THEOREM. *A semilattice* $\mathfrak{S}$ *is equationally compact if and only if the following three conditions are satisfied*:

(i) $\mathfrak{S}$ *is join-complete*;

(ii) *any chain* C *in* $\mathfrak{S}$ *has a meet*;

(iii) *if* $a \in S$, *and* C *is a chain in* $\mathfrak{S}$, *then*

$$a \vee \bigwedge(x \mid x \in C) = \bigwedge(a \vee x \mid x \in C).$$

COROLLARY. *A* 1-*equationally compact semilattice is equationally compact.*

2. A *semilattice* $\mathfrak{S} = \langle S; \vee \rangle$ is a set S with a binary operation $\vee$, which is idempotent, commutative, and associative. A partial ordering $\leqslant$ is defined on S by

$$x \leqslant y \text{ iff } x \vee y = y.$$

$\mathfrak{S}$ is *join-complete* if any set $H \subseteq S$ has a least upper bound, denoted by $\bigvee(x \mid x \in H)$. If H has a greatest lower bound, it will be denoted by $\bigwedge(x \mid x \in H)$. A *chain* C *in* $\mathfrak{S}$ is a subset C of S such that for any $a, b \in C$ we have $a \leqslant b$ or $b \leqslant a$.

Let $X = \{x_i \mid i \in I\}$ (the set of "unknowns"). An *equation in* X *over* $\mathfrak{S}$ is an expression of the form

$$p = q, \tag{1}$$

(1) Result announced in Notices of the American Mathematical Society 15 (1968), p. 196.

where p and q are expressions of one of the following three types:

$$(2) \qquad a, \quad x_{i_0} \vee \ldots \vee x_{i_{n-1}}, \quad a \vee x_{i_0} \vee \ldots \vee x_{i_{n-1}},$$

where n is an arbitrary integer, $a \in S$, and $i_0, \ldots, i_{n-1} \in I$.

A *solution* of (1) is a map $\varphi: I \to S$ such that if x_i is substituted by $i\varphi$, then the two sides of (1) yield the same element of S.

Let Σ be an arbitrary set of equations in X over $\mathfrak{S}$. Σ is *solvable* if a single map $I \to S$ is a solution for all equations in Σ; and Σ is *locally solvable* if every finite subset of Σ is solvable.

A semilattice $\mathfrak{S}$ is *equationally compact* if for *any* set X, any locally solvable set Σ of equations in X over $\mathfrak{S}$ is solvable. If this holds for sets X of cardinality 1, $\mathfrak{S}$ is 1-*equationally compact.*

All these concepts are specialized from the concepts of [2], utilizing the special properties of semilattices.

3. Let $\mathfrak{S}$ be equationally compact. We prove that (i)-(iii) hold in $\mathfrak{S}$.

Consider the set Σ_0 of equations

$$a \vee x = x$$

for all $a \in S$. If Σ_0^+ is a finite subset of Σ_0, $\Sigma_0^+ \neq \emptyset$, then it has a solution ($x =$ the join of the a's that occur in Σ_0^+), hence Σ_0 is locally solvable, and hence solvable. A solution $x = 1$ is the largest element of $\mathfrak{S}$. Thus $\mathfrak{S}$ has a 1.

Now let $H \subseteq S$ and

$$U = \{u \mid u \in S,\ u \geqslant h \text{ for all } h \in H\}.$$

Then $U \neq \emptyset$, since $1 \in U$. Consider the set Σ_1 of equations

$$a \vee x = x, x \vee u = u$$

for all $a \in H$, $u \in U$. Again Σ_1 is locally solvable. Thus Σ_1 is solvable and the solution will be $\bigvee(x \mid x \in H)$. This verifies (i).

To verify (ii) let C be a chain and Σ_2 be the set of equations

$$x \vee c = c$$

for all $c \in C$. Again Σ_2 is locally solvable (if Σ_2^+ is a finite subset of Σ_2, then a solution is the *smallest* c that occurs in some equation in Σ_2^+) and a solution will be a lower bound t for C. Thus the set D of *all* lower bounds of C is non-void, and we can consider Σ_3, the set

$$d \vee x = x, \quad x \vee c = c$$

for all $d \in D$, $c \in C$, and the solution of this is $\bigwedge(x \mid x \in C)$.

Finally, let $a \in S$, and C a chain and put $c = \bigwedge(x \mid x \in C)$ (this exists by (ii)). Since $C_1 = \{a \vee x \mid x \in C\}$ is again a chain, by (ii) $\bigwedge(y \mid y \in C_1) = d$ exists. We have to prove that $a \vee c = d$. Of course, $a \vee c \leqslant d$. Now take the set Σ_4 of equations

$$a \vee x = d \vee x, \quad x \vee b = b$$

for all $b \in C$. If Σ_4^+ is a finite subset of Σ_4, then again the smallest of the b is a solution; hence Σ_4 has a solution c_1. Then $a \vee c_1 = d \vee c_1$, hence $a \vee c_1 \geqslant d$. But $c_1 \vee b = b$ for all $b \in C$, and so $c_1 \leqslant c$. Thus $a \vee c \geqslant a \vee c_1 \geqslant d$, completing the proof of (iii).

4. To prepare the proof of sufficiency we state four lemmas. A subset D of S is *downward directed* if for $x, y \in D$ there exists $z \in D$ with $z \leqslant x$ and $z \leqslant y$.

LEMMA 1. *Condition* (ii) *implies that* $\bigwedge(x \mid x \in D)$ *exists for any downward directed set* D.

LEMMA 2. *Conditions* (ii) *and* (iii) *imply* $a \vee \bigwedge(x \mid x \in D) = \bigwedge(a \vee x \mid x \in D)$ *for any downward directed set* D.

The proofs of Lemmas 1 and 2 follow the well-known pattern (see e.g. [1], Appendix 2) and will therefore be omitted.

A solution is an element of S^I. Thus we have a natural partial ordering for them: the pointwise ordering. The following lemma is crucial:

LEMMA 3. *Let* $\mathfrak{S}$ *satisfy* (i)-(iii), $p = q$ *be an equation, and* $K \subseteq S^I$ *be a downward directed set of solutions for* $p = q$. *Then* $t = \bigwedge(k \mid k \in K)$ *exists and it is a solution for* $p = q$.

Proof. By Lemma 1 (ii) holds for directed sets in $\mathfrak{S}$, hence in $\mathfrak{S}^I$. Thus k exists.

Let $K = \{c_\lambda \mid \lambda \in \Lambda\}$; $c_\lambda(i)$ will denote the i-th component of c_λ. Then $t(i) = \bigwedge(c_\lambda(i) \mid \lambda \in \Lambda)$.

Let $p = a \vee x_{i_0} \vee \ldots \vee x_{i_{n-1}}$. Then $p(t) = a \vee t(i_0) \vee \ldots \vee t(i_{n-1}) = a \vee \vee \bigwedge\big(c_\lambda(i_0) \mid \lambda \in \Lambda\big) \vee \ldots \vee \bigwedge\big(c_\lambda(i_{n-1}) \mid \lambda \in \Lambda\big) = \bigwedge\big(a \vee c_{\lambda_0}(i_0) \vee \ldots \vee c_{\lambda_{n-1}}(i_{n-1}) \mid \lambda_0, \ldots \ldots, \lambda_{n-1} \in \Lambda\big) = \bigwedge\big(a \vee c_\lambda(i_0) \vee \ldots \vee c_\lambda(i_{n-1}) \mid \lambda \in \Lambda\big) = \bigwedge\big(p(c_\lambda) \mid \lambda \in \Lambda\big)$ where the third and the fourth equalities hold, since K is downward directed and Lemma 2 can be repeatedly applied.

Similarly, $q(t) = \bigwedge\big(q(c_\lambda) \mid \lambda \in \Lambda\big)$ and so $p(t) = q(t)$, which was to be proved.

If p or q are of the form a or $x_{i_0} \vee \ldots \vee x_{i_{n-1}}$, the computation proceeds similarly.

LEMMA 4. *Let* $\mathfrak{S}$ *satisfy* (i) *and let* K *be a set of solutions for* $p = q$. *Then* $t = \bigvee(k \mid k \in K)$ *is a solution for* $p = q$.

Proof. Again, we take $p = a \vee x_{i_0} \vee \ldots \vee x_{i_{n-1}}$, $q = b \vee x_{j_0} \vee \ldots \vee x_{j_{m-1}}$. Then for any $k \in K$, we get

$$p(t) = a \vee t(i_0) \vee \ldots \vee t(i_{n-1}) \geqslant a \vee k(i_0) \vee \ldots \vee k(i_{n-1})$$
$$= b \vee k(j_0) \vee \ldots \vee k(j_{m-1}) = q(k),$$

hence $p(t) \geqslant \bigvee \big(q(k) \mid k \in K\big) = q(t)$. Similarly, $q(t) \geqslant p(t)$, hence $p(t) = q(t)$.

5. Now we are ready to prove the sufficiency. Let us assume that (i)-(iii) hold for $\mathfrak{S}$, and let Σ be a locally solvable set of equations. Let K_{Σ^+} be the set of solutions for a finite subset Σ^+ of Σ. By assumption $\Sigma^+ \neq \emptyset$. Set $t_{\Sigma^+} = \bigvee(k \mid k \in \Sigma^+)$. Then, by Lemma 4, $t_{\Sigma^+} \in K_{\Sigma^+}$.

Set $K = \{t_{\Sigma^+} \mid \emptyset \neq \Sigma^+ \subseteq \Sigma, \Sigma^+ \text{ is finite}\}$. Then K is downward directed, since $\Sigma^+ \supseteq \Sigma^{++}$ implies $t_{\Sigma^+} \leqslant t_{\Sigma^{++}}$.

Set $t = \bigwedge(k \mid k \in K)$. By Lemma 3, t is a solution for Σ, completing the proof of the Theorem.

6. In section 3 all sets of equations we considered contained only one "unknown", hence the Corollary is true.

7. The following statement follows easily from the Theorem:

An equationally complete semilattice $\mathfrak{S}$ is either a lattice (as a partially ordered set) or there are elements $a, b \in S$ such that a and b have no lower bound. In the latter case $\mathfrak{S}$ can be made a lattice by adjoining a 0. In both cases the lattice is a complete lattice satisfying (iii).

Conversely, if $\mathfrak{L}$ is a complete lattice satisfying (iii) in which each element contains an atom, then $\mathfrak{L} - \{0\}$ is an equationally compact semilattice.

8. A trivial application of the Theorem yields the following statement:

Let $\mathfrak{L}$ be an equationally compact lattice. Then $\mathfrak{L}$ is complete, furthermore (iii) and its dual hold for $\mathfrak{L}$.

The converse is not, however, true, see [3].

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