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A REMARK ON THE H^1-BMO DUALITY IN PRODUCT DOMAINS

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This note deals with the following problem:

Given $g \in L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$ satisfying a Carleson condition, prove that g defines a linear continuous functional on the Hardy space $H^1(\boldsymbol{R}\times\boldsymbol{R})$, defining $H^1(\boldsymbol{R}\times\boldsymbol{R})$ in terms of the nontangential maximal function.

Indeed, duality, atomic decomposition and related topics are usually handled by means of the area function. At first glance, to get these results in terms of maximal functions implies to deal with L^∞-valued operators. The duality proof given here handles this difficulty by following an idea of J. M. Wilson. In [6], Wilson combines the notion of tent (see [4]) and a lemma by K. G. Merryfield (see [5]) in order to obtain the atomic decomposition of functions in $H^p(\boldsymbol{R}\times\boldsymbol{R})$ by means of maximal functions. See [3] for a survey on the subject.

Let $f \in H^1(\boldsymbol{R}\times\boldsymbol{R})$. We start by recalling the A. P. Calderón representation of f (see [1], p. 220). Let $\psi \in C_0^\infty(\boldsymbol{R})$ be a real, radial function such that

$$\operatorname{supp}\psi \subset \{|x| \leqslant 1\} \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-\pi|\theta|}\hat{\psi}(\theta)\,d\theta = -1/\pi.$$

Given $s > 0$, set

$$\psi_s(x) = \frac{1}{s}\psi\left(\frac{x}{s}\right).$$

Then

$$f = \int_{\boldsymbol{R}^2_+\times\boldsymbol{R}^2_+} \frac{\partial^2 u}{\partial t_1 \partial t_2}(y, t)\psi_{t_1}(x_1-y_1)\psi_{t_2}(x_2-y_2)\,dy\,dt, \tag{1}$$

where u stands for the double Poisson transform of f, $y = (y_1, y_2)$, $t = (t_1, t_2)$.

Following Wilson, for each $k \in \boldsymbol{Z}$ let

$$E_k = \{x \in \boldsymbol{R}^2\colon A_\infty^{(\alpha)}(u)(x) > 2^k\}, \quad \text{given } \alpha > 0 \text{ large},$$
$$F_k = \{x \in \boldsymbol{R}^2\colon M(\chi_{E_k})(x) > \varepsilon\}, \quad \text{given } 0 < \varepsilon < 1,$$

where

$$A_\infty^{(\alpha)}(u)(x) = \sup_{|x_i - y_i| < \alpha t_i} |u(y, t)|$$

is the nontangential maximal function and M denotes the strong maximal function of Jessen, Marcinkiewicz and Zygmund, χ_{E_k} is the characteristic function of E_k, E_k being a subset of F_k of finite Lebesgue measure. Moreover, since M is of weak type $(L\log L, L^1)$ in $\boldsymbol{R}^2$, there exists $C = C(\varepsilon) > 0$ such that $|F_k| \leqslant C|E_k|$, where $|\ |$ denotes the Lebesgue measure in $\boldsymbol{R}^2$.

Now, given any subset $\Omega \subset \boldsymbol{R}^2$, let

$$\hat{\Omega} = \{(y, t) \in \boldsymbol{R}^2_+ \times \boldsymbol{R}^2_+ :\ R_{y,t} \subset \Omega\},$$

where $R_{y,t}$ is the rectangle with sides parallel to the coordinate axes, centered at $y \in \boldsymbol{R}^2$, with side lengths t_1 and t_2. $\hat{\Omega}$ is called the *tent* over the set Ω (see [4]).

If $T_k = \hat{F}_k \backslash \hat{F}_{k+1}$, the family $\{T_k\}_{k\in\boldsymbol{Z}}$ is a disjoint covering of $\operatorname{supp}(u)$. This comes from the following fact which is easy to prove (see [4]). Given $\lambda > 0$,

$$\{(y, t) \in \boldsymbol{R}^2_+ \times \boldsymbol{R}^2_+ :\ |u(y, t)| > \lambda\} \subset \{x \in \boldsymbol{R}^2 :\ A^{(\alpha)}_\infty(u)(x) > \lambda\}^{\hat{}}\,.$$

Thus, the integral in (1) can be written as

$$\sum_{k\in\boldsymbol{Z}} \int_{\boldsymbol{R}^2_+ \times \boldsymbol{R}^2_+} \chi_{T_k}(y, t) \frac{\partial^2 u}{\partial t_1 \partial t_2}(y, t) \psi_{t_1}(x_1 - y_1) \psi_{t_2}(x_2 - y_2)\, dy\, dt.$$

This is essentially the atomic decomposition of f (see [2]).

Now, let $g \in L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$ be a function satisfying the Carleson condition. Particularly,

$$|(\psi_{t_1}\psi_{t_2} * g)(y)|^2 \frac{dy\,dt}{t_1 t_2}$$

is a Carleson measure for the given ψ. We can write

$$\int_{\boldsymbol{R}^2} f\bar{g}\,dx = \sum_{k\in\boldsymbol{Z}} \int_{\boldsymbol{R}^2} \int_{\boldsymbol{R}^2_+ \times \boldsymbol{R}^2_+} \chi_{T_k}(y, t) t_1 t_2 \frac{\partial^2 u}{\partial t_1 \partial t_2}(y, t) \psi_{t_1}(x_1 - y_1) \psi_{t_2}(x_2 - y_2) \bar{g}(x) \frac{dy\,dt}{t_1 t_2}$$

or

$$\int_{\boldsymbol{R}^2} f\bar{g}\,dx = \sum_{k\in\boldsymbol{Z}} \int_{\boldsymbol{R}^2_+ \times \boldsymbol{R}^2_+} \chi_{T_k}(y, t) t_1 t_2 \frac{\partial^2 u}{\partial t_1 \partial t_2}(y, t) (\psi_{t_1}\psi_{t_2} * \bar{g})(y) \frac{dy\,dt}{t_1 t_2}.$$

The Cauchy–Schwarz inequality yields

$$\Big|\int_{\boldsymbol{R}^2} f\bar{g}\,dx\Big| \leqslant \sum_{k\in\boldsymbol{Z}} \left(\int_{T_k} t_1^2 t_2^2 \left| \frac{\partial^2 u}{\partial t_1 \partial t_2}(y, t) \right|^2 \frac{dy\,dt}{t_1 t_2} \right)^{1/2} \left(\int_{\hat{F}_k} |(\psi_{t_1}\psi_{t_2} * g)(y)|^2 \frac{dy\,dt}{t_1 t_2} \right)^{1/2}$$

or

$$\Big|\int_{\boldsymbol{R}^2} f\bar{g}\,dx\Big| \leqslant C \sum_{k\in\boldsymbol{Z}} \left(\int_{T_k} t_1 t_2 |\partial^2 u(y, t)|^2\, dy\, dt \right)^{1/2} |E_k|^{1/2}, \tag{2}$$

where ∂^2 denotes the formal vector

$$\left(\frac{\partial^2}{\partial t_1 \partial t_2}, \frac{\partial^2}{\partial t_1 \partial y_2}, \frac{\partial^2}{\partial t_2 \partial y_1}, \frac{\partial^2}{\partial y_1 \partial y_2}\right)$$

and $C = C(g)$.

Suppose we prove that

$$\int_{T_k} t_1 t_2 |\partial^2 u(y, t)|^2 dy dt \leqslant C 2^{2k} |E_k|, \tag{3}$$

where C does not depend on k. Then (2) reduces to

$$\left| \int_{\mathbf{R}^2} f\bar{g} dx \right| \leqslant C \sum_{k \in \mathbf{Z}} 2^k |E_k|.$$

Now, set $H_j = E_j \backslash E_{j+1}$. Then $E_k = \bigcup_{j \geqslant k} H_j$, which is a disjoint union. Thus

$$\begin{aligned}
\sum_{k\in \mathbf{Z}} 2^k |E_k| &= \sum_{k \in \mathbf{Z}} 2^k \sum_{j \geqslant k} |H_j| \\
&\leqslant \sum_{k \in \mathbf{Z}} 2^k \sum_{j \geqslant k} 2^{-j} \int_{H_j} A_\infty^{(\alpha)}(u)(x) dx = \sum_{j \in \mathbf{Z}} 2^{-j} \Big(\int_{H_j} A_\infty^{(\alpha)}(u)(x) dx \sum_{k \leqslant j} 2^k \Big) \\
&= 2 \sum_{j \in \mathbf{Z}} \int_{H_j} A_\infty^{(\alpha)}(u)(x) dx = 2 \|A_\infty^{(\alpha)}(u)\|_{L^1} \approx \|f\|_{H^1}.
\end{aligned}$$

Thus, it only remains to prove (3). This is inequality (2) in [6], p. 205. So, we will only outline the proof. It uses the following lemma, which is a particular case of Theorem 2.1 in [5], p. 665:

LEMMA. *Let $G \subset \mathbf{R}^2$ be a set with finite Lebesgue measure. Let u be a 2-harmonic function defined in $\mathbf{R}_+^2 \times \mathbf{R}_+^2$. Suppose that, for some $\lambda > 0$, $A_\infty^{(\alpha)}(u)(x) \leqslant \lambda$ if $x \in G$. Then there exists $C > 0$ not depending on λ such that*

$$\int_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} t_1 t_2 |\partial^2 u(y, t)|^2 |(\varphi_{t_1} \varphi_{t_2} * \chi_G)(y)|^2 dy \leqslant C \lambda^2 |G|,$$

where $\varphi \in C_0^\infty(\mathbf{R})$, $\varphi \geqslant 0$, $\operatorname{supp} \varphi \subset [-1, 1]$, $\varphi \geqslant 1/2$ on $[-1/2, 1/2]$ and $\int \varphi dx = 1$.

We will use this lemma with $G = F_k \backslash E_{k+1}$, given $k \in \mathbf{Z}$ fixed. So $\lambda = 2^{k+1}$. Thus,

$$\int_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} t_1 t_2 |\partial^2 u(y, t)|^2 |(\varphi_{t_1} \varphi_{t_2} * \chi_G)(y)|^2 dy dt \leqslant C 2^{2k} |E_k|.$$

In order to get (3) it suffices to show that, for some $\delta > 0$,

$$(\varphi_{t_1} \varphi_{t_2} * \chi_G)(y) > \delta \quad \text{when } (y, t) \in T_k.$$

In fact, given $(y, t) \in T_k$, this means that $R_{y,t} \subset F_k$ but there exists $x \in R_{y,t}$ such that $x \notin F_{k+1}$. Thus $|R_{y,t} \cap F_k| = |R_{y,t}|$, but

$$|R_{y,t} \cap E_{k+1}| < \varepsilon |R_{y,t}|.$$

Consequently,

$$|R_{y,t} \cap G| > (1-\varepsilon)|R_{y,t}| = (1-\varepsilon)t_1 t_2.$$

Now, according to the properties of φ,

$$(\varphi_{t_1 t_2} * \chi_G)(y) = \int_G \frac{1}{t_1 t_2} \varphi\left(\frac{y_1 - z_1}{t_1}\right)\varphi\left(\frac{y_2 - z_2}{t_2}\right) dz_1 dz_2$$

$$\geqslant \frac{1}{4t_1 t_2} \int_{R_{y,t} \cap G} dz_1 dz_2 = \frac{1}{4} \frac{|R_{y,t} \cap G|}{|R_{y,t}|} > \frac{1-\varepsilon}{4}.$$

This completes the proof.

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