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ON SOME OPERATION ON SETS OF MAPPINGS

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The present paper is a continuation of the paper(1) containing some considerations dealing with the topological theory of atlases. In that paper a condition was given which may be treated as a new axiom of the theory of atlases. This condition allows to introduce the concept of a structure being a generalization of the notion of a differential structure on a manifold. Generalization goes in two directions: first, it replaces the assumption that the space of parameters is a Euclidean space or a half-space by the assumption that the space of parameters is an arbitrary topological space; and second, it considers an arbitrary family $\mathscr{C}$ of functions acting in the space of parameters instead of the family of all diffeomorphisms which transform open sets onto open sets. In the present paper, the parameter space and the topological space having structure of a generalized differential manifold is supposed to be an element of the given class $\boldsymbol{K}$ of topological spaces. For example, $\boldsymbol{K}$ may be the class of all Hausdorff spaces, the class of all regular spaces, the class of all completely regular spaces, etc. The essential part play, in the present paper, hereditary classes of topological spaces. Concerning these classes we also make assumption that they are full. For every such class $\boldsymbol{K}$, for an arbitrary topological space P belonging to the class $\boldsymbol{K}$ and for any set $\mathscr{C}$ of functions acting in P, we consider the subset $\mathscr{C}_{\boldsymbol{K},P}$ of $\mathscr{C}$ such that, for any space X of the class $\boldsymbol{K}$, the set of all atlases of the class C on the space X, with respect to the topological space P (as a parameter space), remains non-changed.

For any function f and for any set B, D_f and $f^{-1}[B]$ will denote the domain of f and the counter-image of B given by the function f. If $A \subset D_f$ we denote by $f \mid A$ and $f[A]$ the restriction of f to A and the f-image of A, respectively. For given functions g and f, we denote by $g \circ f$ the composition of f and g, i. e., the function whose domain is the set $f^{-1}[D_g]$

(1) W. Waliszewski, *On the transitivity of some relations between atlases*, Colloquium Mathematicum 20 (1969), p. 259-263.

and such that $(g \circ f)(x) = g(f(x))$ for $x \in f^{-1}(D_g]$. For a given set A, id_A denotes the identity function on A. For a topological space X and for a set A of its points, we denote by $X \mid A$ the topological space induced on A by the space X.

A function f such that D_f is open in the topological space X, $f[D_f]$ is open in the topological space P, and f is a homeomorphism of $X \mid D_f$ onto $P \mid f[D_f]$ will be called a *map* of the space X with respect to P. The set F of all maps of the space X with respect to the space P is called an *atlas* of the class $\mathscr{C}$ of the space X with respect to the space P ($\mathscr{C}$ is here an arbitrary set of functions f such that the set $D_f \cup f[D_f]$ is included in the set of all points of P) if it satisfies the following conditions:

(0.1) $\bigcup_{f \in F} D_f$ is identical with the set of all points of X,

(0.2) $g \circ f^{-1} \in C$ for any $g, f \in F$.

The set of all atlases of the class $\mathscr{C}$ of the space X with respect to the space P is denoted by $\mathfrak{A}(C, X, P)$. Now we may define the set $\mathscr{C}_{K,P}$ precisely as the intersection of all $\mathscr{C}'$ contained in $\mathscr{C}$ such that

(0.3) for every topological space X of the class K the equality

$$\mathfrak{A}(C', X, P) = \mathfrak{A}(\mathscr{C}, X, P) \tag{1}$$

holds.

Equality (1) says that the set of all atlases of the class $\mathscr{C}'$ of the topological space X with respect to the topological space $\mathscr{P}$, treated as a parametric space, is equal to a similar set of all atlases corresponding to the class $\mathscr{C}$. Let us remark that if the function f acts in P, then f need not be continuous, but only its domain and the set of all values must be some sets of points of P.

1. The operation that assigns to every class $\mathscr{C}$ of mappings acting in P the class $\mathscr{C}_{K,P}$ removes mappings that are superfluous in the definition of the class of all atlases in a given category of topological spaces. We prove that

1.1. *For every class K of topological spaces, for every P of K, and for every set $\mathscr{C}$ of mappings acting in P the equality*

$$\mathfrak{A}(\mathscr{C}_{K,P}, X, P) = \mathfrak{A}(\mathscr{C}, X, P) \tag{2}$$

is fulfilled for every topological space X of the class K. Thus, $\mathscr{C}_{K,P}$ is the minimal set among the subsets $\mathscr{C}'$ of $\mathscr{C}$ satisfying (*). *The set $\mathscr{C}_{K,P}$ is contained in the set of all homeomorphisms of open subspaces of P onto open subspaces of P.*

Proof. Let $X \in K$ and $F \in \mathfrak{A}(\mathscr{C}, X, P)$. Then $\bigcup_{f \in F} D_f$ is identical with the set of all points of X. Take $g, f \in F$. Then $g \circ f^{-1} \in \mathscr{C}$. Consider an arbitrary subset $\mathscr{C}'$ of $\mathscr{C}$ fulfilling (1). From the definition of the set $\mathfrak{A}(\mathscr{C}', X, P)$ of all atlases of the class $\mathscr{C}'$ of the topological space X with respect to the topological space P (see [1]), it follows that $g\circ f^{-1}\in\mathscr{C}'$. Thus, $g\circ f^{-1}\in\mathscr{C}_{K,P}$. Therefore the inclusion

$$\mathfrak{A}(\mathscr{C}, X, P) \subset \mathfrak{A}(\mathscr{C}_{K,P}, X, P)$$

is satisfied. The inverse inclusion follows from the fact that $\mathscr{C}_{K,P}$ is contained in $\mathscr{C}$. Then for every topological space X of the class $\boldsymbol{K}$ equality (2) holds. The last part of the statement immediately follows from the fact that condition (*) is fulfilled by the set $\mathscr{C}'$ of all $\varphi\in C$ which are homeomorphisms of open subspaces of P.

2. Now we shall give a characterization of the set $\mathscr{C}_{K,P}$ by taking a certain decomposition space.

Let X and Y be topological spaces and f be a one-to-one function, the domain D_f of f be a set of points of X, and the set $f[D_f]$ of all values of f be a set of points of Y. To assure disjointness of the set of all points of X and the set of all points of Y, consider the spaces X_0 and Y_1: the set of all open sets of X_0 is the set of all sets $A\times\{0\}$, where A is open in X and, analogously, the set of all open sets of Y_1 is the set of all sets $B\times\{1\}$, where B is open in Y. Let us set

(3) $\quad i_0(x) = (x, 0)$ for x of X and $i_1(y) = (y, 1)$ for y of Y.

The mappings i_0 and i_1 are homeomorphisms of X onto X_0 and Y onto Y_1, respectively. Setting

$$f^* = i_1\circ f\circ i_0^{-1}$$

we may define some equivalence in the following manner: $p\equiv q$ if and only if

(2.0) *p and q are points of the disjoint union $X_0\oplus Y_1$ of the topological spaces X_0 and Y_1 and there is fulfilled at least one of three cases:*

(2.0.1) $p = q$,

(2.0.2) *p belongs to the domain of f^* and $q = f^*(p)$,*

(2.0.3) *q belongs to the domain of f^* and $p = f^*(q)$.*

Let us denote by $\mathrm{gl}(X, Y, f)$ the topological space $(X_0\oplus Y_1)/\equiv$. We shall prove the following lemma:

2.1. *If the domain of f and the set of all values of f are open in the topological spaces X and Y, respectively, and f is a homeomorphism of $X\mid D_f$ onto $Y\mid f[D_f]$, then there exist homeomorphisms f_0 and f_1 such that*

(2.1.1) *the domains D_{f_0} and D_{f_1} of mappings f_0 and f_1, respectively, are open in $\mathrm{gl}(X, Y, f)$, and $D_{f_0}\cup D_{f_1}$ is identical with the set of all points of $\mathrm{gl}(X, Y, f)$,* (2.1.2) f_0 *and* f_1 *are homeomorphisms of* $\mathrm{gl}(X, Y, f) \mid D_{f_0}$ *and* $\mathrm{gl}(X, Y, f) \mid D_{f_1}$ *onto* X *and* Y, *respectively*,

(2.1.3) $f_1 \circ f_0^{-1} = f$.

Proof. Let $k(p)$ be the equivalence class of the relation $\equiv$ containing the point p of $X_0 \oplus Y_1$. Put

$$Z = \mathrm{gl}(X, Y, f).$$

Then k is a continuous mapping of $X_0 \oplus Y_1$ onto Z. First of all we remark that for every set G of points of the topological space X we have

$$k^{-1}[k_0[G]] = G \times \{0\} \cup f[D_f \cap G] \times \{1\}.$$

Then, if G is open in X, we state that $k^{-1}[k_0[G]]$ is open in $X_0 \oplus Y_1$. Therefore, $k_0[G]$ is open in Z. Hence it follows that k_0 is an open mapping of X into Z. Similarly, we verify that k_1 is an open mapping of Y into Z. To prove that k_0 and k_1 are one-to-one mappings let us take any points x and y of X such that $k_0(x) = k_0(y)$. Setting $p = i_0(x)$ and $q = i_0(y)$ we get $k(p) = k(q)$, where p and q are points of X_0. Then $p \equiv q$. From the fact that the set of all values of f^* is contained in the set of all points of Y_1, which is disjoint with the set of all points of X_0, it follows immediately that neither (2.0.2) nor (2.0.3) is fulfilled. Then it must be $p = q$. Thus, k_0 and, analogously, k_1 are homeomorphisms of X and Y onto the topological space induced by Z in the k_0-image of the set of all points of X and, respectively, by the analogous k_1-image of the set of all points of Y. On the other hand, these k_i-images ($i = 0, 1$) are open in Z and their union covers Z. Setting $f_0 = k_0^{-1}$ and $f_1 = k_1^{-1}$, we obtain mappings fulfilling conditions (2.1.1), (2.1.2) and (2.1.3).

The lemma which we have proved above is helpful in the proof of the theorem giving some new definition of the class $\mathscr{C}_{K,P}$. Before formulation of the theorem let us recall that the class $\boldsymbol{K}$ of topological spaces is full if and only if for every topological space all spaces homeomorphic to it belong to $\boldsymbol{K}$.

2.2. THEOREM. *If* $\boldsymbol{K}$ *is an arbitrary full and hereditary class (with respect to open subspaces), if* P *is a topological space of the class* $\boldsymbol{K}$, *and if* $\mathscr{C}$ *is any set of functions acting in* P, *then the set* $\mathscr{C}_{K,P}$ *is the set of all mappings* φ *of* $\mathscr{C}$ *fulfilling the following conditions*:

(2.2.1) *the domain* D_φ *of* φ *and the set* $\varphi[D_\varphi]$ *are open in* P,

(2.2.2) φ *is a homeomorphism of* $P \mid D_\varphi$ *onto* $P \mid \varphi[D_\varphi]$,

(2.2.3) $\varphi^{-1} \in \mathscr{C}$ *and there exist sets* A *and* B *open in* P *such that* id_A, $\mathrm{id}_B \in C$, $D_\varphi \subset A$, $\varphi[D_\varphi] \subset B$ *and* $\mathrm{gl}(P \mid A, P \mid B, \varphi)$ *is a topological space of the class* $\boldsymbol{K}$.

Proof. Consider the set $\mathscr{C}'$ of all mappings $\varphi \in \mathscr{C}$ fulfilling conditions (2.2.1), (2.2.2) and (2.2.3), and denote by Z the topological space $\mathrm{gl}(X, Y, \varphi)$, where

(4) $$X = P \mid A, \qquad Y = P \mid B,$$

and A as well as B are sets appearing in (2.2.3). From Lemma 2.1 it immediately follows that there exist mappings f_0 and f_1 fulfilling conditions (2.1.1), (2.1.2) and (2.1.3). Setting $F = \{f_0, f_1\}$ we see that

$$f_0 \circ f_0^{-1} = \mathrm{id}_A, \quad f_1 \circ f_1^{-1} = \mathrm{id}_B \quad \text{and} \quad f_1 \circ f_0^{-1} = \varphi.$$

Then, by (2.2.3), F belongs to $\mathfrak{A}(\mathscr{C}, Z, P)$. From Lemma 1.1 we get $F \in \mathfrak{A}(\mathscr{C}_{K,P}, Z, P)$. Then $\varphi \in \mathscr{C}_{K,P}$. To prove that $\mathscr{C}_{K,P}$ is contained in $\mathscr{C}'$, take an arbitrary topological space T of the class K. It is sufficient to check the equality

$$\mathfrak{A}(\mathscr{C}', T, P) = \mathfrak{A}(\mathscr{C}, T, P).$$

It is evident that the set $\mathfrak{A}(\mathscr{C}', T, P)$ is contained in $\mathfrak{A}(\mathscr{C}, T, P)$. Now take an arbitrary F belonging to the set $\mathfrak{A}(\mathscr{C}, T, P)$ and arbitrary maps f and g of the atlas F. First, we shall prove that the topological space $T \mid (D_f \cup D_g)$ is homeomorphic to the topological space $\mathrm{gl}(X, Y, \varphi)$, where X and Y are given by (4), $A = f[D_f]$, $B = g[D_g]$ and $\varphi = g \circ f^{-1}$. Consider any $x \in D_f \cap D_g$. Setting $p = i_0(f(x))$ and $q = i_1(g(x))$, where i_0 and i_1 are defined by (3), we have

$$\varphi^*(p) = i_1\big(\varphi(i_0^{-1}(p))\big) = i_1\Big(g\big(f^{-1}(i_0^{-1}(p))\big)\Big) = i_1\big(g(x)\big) = q.$$

Then $p \equiv q$. Hence it follows that the definition of the function h given by the formula

(5) $$h(x) = \begin{cases} k\big(i_0(f(x))\big) & \text{for } x \in D_f, \\ k\big(i_1(g(x))\big) & \text{for } x \in D_g \end{cases}$$

is correct, where the meaning of $k(p)$ is the same as in the proof of Lemma 2.1, i. e. $k(p)$ is an equivalence class of the relation $\equiv$ containing the point p. From formula (5) it follows that the function h is a continuous mapping of the topological space $T \mid (D_f \cup D_g)$ into Z. By an easy verification we see that the function h is a one-to-one mapping of the space $T \mid (D_f \cup D_g)$ onto Z. The continuity of the function h^{-1} follows from the fact that, for any open set G of the topological space $T \mid (D_f \cup D_g)$, we have

$$k^{-1}\big[h[G]\big] = i_0\big[f[D_f \cap G]\big] \cup i_1\big[g[D_g \cap G]\big].$$

The hypothesis that the class K is hereditary with respect to open subspaces yields the fact that $T \mid (D_f \cup D_g)$ is an element of K. And from the fullness of K we infer that the topological space Z belongs to K. The proof is completed.

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