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TWO REMARKS ON THE KHINTCHINE-KAHANE INEQUALITY

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Let $(X, \|\cdot\|)$ be a normed space and let $\varepsilon_1, \ldots, \varepsilon_{2n}$ be a Bernoulli sequence of independent random variables defined on the probability space $(\Omega, \mathfrak{M}, P)$ (i.e. $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$ for $i = 1, \ldots, 2n$). The Khintchine-Kahane inequality states that for every $p_1, p_2 > 0$ there exists a constant C_{p_1,p_2} such that, for any $a_i \in X$ $(i = 1, \ldots, n)$,

$$C_{p_1,p_2}\Big(\int_\Omega \Big\| \sum_{i=1}^n \varepsilon_i a_i \Big\|^{p_1} dP\Big)^{1/p_1} \geqslant \Big(\int_\Omega \Big\| \sum_{i=1}^n \varepsilon_i a_i \Big\|^{p_2} dP\Big)^{1/p_2}. \tag{1}$$

Recently, a new proof of inequality (1) was given by C. Borell (see [3]). In this paper we consider only the case $p_1 = 1$, $p_2 = 2$ of the Khintchine-Kahane inequality. We shall prove it in a simple way with $C_{1,2} = \sqrt{3}$. This constant is, as far as we know, the best known by now. It is still an open problem what is the best constant $C_{1,2}$. In a special case, where the space X is a real line, this problem was solved by Szarek [4] and later by Haagerup [1]. The best constant found by them is equal to $\sqrt{2}$. In the second part of the paper we shall give some strengthening of the Khintchine-Kahane inequality.

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THEOREM 1 (cf. Kahane [2]). *For every normed space $(X, \|\cdot\|)$ and for arbitrary elements $a_1, \ldots, a_n$ of X the inequality*

$$\sqrt{3}\int_\Omega \Big\| \sum_{i=1}^n \varepsilon_i a_i \Big\| dP \geqslant \Big(\int_\Omega \Big\| \sum_{i=1}^n \varepsilon_i a_i \Big\|^2 dP\Big)^{1/2} \tag{2}$$

holds.

Proof. Given $a_i \in X$ (for $i = 1, \ldots, n$) and $E \subset S = \{1, \ldots, n\}$ we put

$$\eta_E = \sum_{i\in E} \varepsilon_i a_i, \quad \eta'_E = \sum_{i\in E} \varepsilon_{i+n} a_i, \quad \eta_S = \eta, \quad \eta'_S = \eta', \quad \tilde{\varepsilon}_i = \varepsilon_i \varepsilon_{i+n},$$

and

$$\Omega_E = \{\omega \in \Omega: \tilde{\varepsilon}_i(\omega) = 1 \text{ for } i \in E \text{ and } \tilde{\varepsilon}_i(\omega) = -1 \text{ for } i \notin E\}$$
$$= \{\omega \in \Omega: \varepsilon_i(\omega) = \varepsilon_{i+n}(\omega) \text{ for } i \in E \text{ and } \varepsilon_i(\omega) = -\varepsilon_{i+n}(\omega) \text{ for } i \notin E\}.$$

First we prove that

$$\int_\Omega \left\| \frac{\eta+\eta'}{2} \right\| \left\| \frac{\eta-\eta'}{2} \right\| dP \leqslant \Big(\int_\Omega \|\eta\| dP \Big)^2. \tag{3}$$

Indeed, we have

$$\int_\Omega \left\| \frac{\eta+\eta'}{2} \right\| \left\| \frac{\eta-\eta'}{2} \right\| dP = \sum_{E\subset S} \int_{\Omega_E} \|\eta_E\| \, \|\eta_{S\setminus E}\| dP. \tag{4}$$

Now observe that the variables $\varepsilon_1, \ldots, \varepsilon_n, \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_n$ are independent, and hence the variables $\|\eta_E\|$, $\|\eta_{S\setminus E}\|$, and χ_{Ω_E} (depending on disjoint blocks of $\varepsilon_1, \ldots, \varepsilon_n, \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_n$) are also independent. Therefore

$$\int_{\Omega_E} \|\eta_E\| \, \|\eta_{S\setminus E}\| dP = \int_\Omega \chi_{\Omega_E} \|\eta_E\| \, \|\eta_{S\setminus E}\| dP \tag{5}$$
$$= P(\Omega_E) \int_\Omega \|\eta_E\| dP \int_\Omega \|\eta_{S\setminus E}\| dP.$$

Since the variables $\eta_E - \eta_{S\setminus E}$ and $\eta = \eta_E + \eta_{S\setminus E}$ have the same distribution, we obtain

$$\int_\Omega \|\eta_E\| dP \leqslant \frac{1}{2} \int_\Omega [\|\eta_E + \eta_{S\setminus E}\| + \|\eta_E - \eta_{S\setminus E}\|] dP \tag{6}$$
$$= \frac{1}{2} \int_\Omega \|\eta_E + \eta_{S\setminus E}\| dP + \frac{1}{2} \int_\Omega \|\eta_E - \eta_{S\setminus E}\| dP = \int_\Omega \|\eta\| dP.$$

In the same manner we prove that

$$\int_\Omega \|\eta_{S\setminus E}\| dP \leqslant \int_\Omega \|\eta\| dP. \tag{7}$$

Combining (4)-(7) and the fact that $P(\Omega_E) = 1/2^n$ for every $E \subset S$, we get (3).

On the other hand, by the triangle inequality we have

$$2 \left\| \frac{\eta+\eta'}{2} \right\| \left\| \frac{\eta-\eta'}{2} \right\| \geqslant 2 \left[\frac{\|\eta\| - \|\eta'\|}{2} \right] = \frac{1}{2} (\|\eta\|^2 + \|\eta'\|^2) + \|\eta\| \, \|\eta'\|. \tag{8}$$

Integrating both sides of inequality (8) and applying inequality (3) we obtain the assertion of Theorem 1.

THEOREM 2. *Let $n > 2$. For $k \leqslant n$ let us put*

$$\Omega^k = \left\{\omega \in \Omega: \sum_{i=1}^{n} \left| \frac{\varepsilon_i(\omega) - \varepsilon_{i+n}(\omega)}{2} \right| = k\right\}.$$

Then there exists a constant A (independent of n, k, the space X, and the choice of $a_i \in X$ $(i = 1, \ldots, n)$) such that

$$\frac{A}{P(\Omega^k)} \int_{\Omega^k} \|\eta\| \, \|\eta'\| dP \geqslant \int_{\Omega} \|\eta\|^2 dP. \tag{9}$$

In other words: the mean value of the product $\|\eta\| \, \|\eta'\|$ over all systems of signs $[\varepsilon_1, \ldots, \varepsilon_{2n}]$ such that the sequence $[\varepsilon_1, \ldots, \varepsilon_n]$ differs from the sequence $[\varepsilon_{n+1}, \ldots, \varepsilon_{2n}]$ exactly at k places is comparable with $\int_{\Omega} \|\eta\|^2 dP$.

Remarks. 1. Multiplying the inequality (9) by $P(\Omega^k)$ and summing over k, $0 \leqslant k \leqslant n$, we get the Khintchine-Kahane inequality (1) for $p_1 = 1$, $p_2 = 2$ with $C_{1,2} = A$.

2. The assertion of Theorem 2 fails to be true in the case $n = 2$. Take, e.g., $a_1 = a_2 \neq 0$ and $k = 1$.

Proof of Theorem 2. Let $\mathscr{W}$ be the family of all subsets of S having exactly k elements. Since

$$\Omega^k = \bigcup_{E \in \mathscr{W}} \Omega_E,$$

we have

$$\int_{\Omega^k} \|\eta\| \, \|\eta'\| dP = \sum_{E \in \mathscr{W}} \int_{\Omega_E} \|\eta\| \, \|\eta'\| dP. \tag{10}$$

Since $\eta'(\omega) = \eta_E(\omega) - \eta_{S \setminus E}(\omega)$ for $\omega \in \Omega_E$, we get, similarly as in the proof of (5),

$$\int_{\Omega_E} \|\eta'\| \, \|\eta\| dP = \frac{1}{2^n} \int_{\Omega} \|\eta\| \, \|\eta_E - \eta_{S \setminus E}\| dP. \tag{11}$$

Let R denote the set of all permutations of the set S. Choose an arbitrary set $F \in \mathscr{W}$, $\sigma \in R$, and apply the above equality with $E = \sigma(F)$. Then we have

$$\int_{\Omega_{\sigma(F)}} \|\eta\| \, \|\eta'\| dP = \frac{1}{2^n} \int_{\Omega} \|\eta\| \, \|\eta_{\sigma(F)} - \eta_{\sigma(S \setminus F)}\| dP. \tag{12}$$

Averaging (12) over all $\sigma \in R$ we get

$$(13)\qquad \frac{1}{n!}\sum_{\sigma\in R}\int_{\Omega_{\sigma(F)}} \|\eta\|\,\|\eta'\|dP = \frac{1}{n!}\frac{1}{2^n}\sum_{\sigma\in R}\int_{\Omega} \|\eta\|\,\|\eta_{\sigma(F)}-\eta_{\sigma(S\setminus F)}\|dP.$$

Now observe that for each $E \in \mathscr{W}$ we have $E = \sigma(F)$ exactly for $k!(n-k)!$ permutations $\sigma \in R$. Therefore, by (10) and the fact that

$$P(\Omega^k) = \binom{n}{k}\cdot 2^{-n},$$

we can write (13) in the form

$$(14)\qquad \frac{1}{P(\Omega^k)}\int_{\Omega^k} \|\eta\|\,\|\eta'\|dP = \frac{1}{n!}\sum_{\sigma\in R}\int_{\Omega} \|\eta\|\,\|\eta_{\sigma(F)}-\eta_{\sigma(S\setminus F)}\|dP.$$

Of course, it is enough to prove the assertion of Theorem 2 for $k \leqslant [n/2]$. To do this, we need the inequalities

$$(15)\qquad \frac{4}{n!}\sum_{\sigma\in R}\int_{\Omega} \|\eta_{\sigma(E)}\|\,\|\eta_{\sigma(S\setminus E)}\|dP \geqslant 2\int_{\Omega}\|\eta\|^2dP - \frac{2}{P(\Omega^k)}\int_{\Omega^k}\|\eta\|\,\|\eta'\|dP$$

and

$$(16)\qquad \frac{1}{P(\Omega^k)}\int_{\Omega^k} \|\eta\|\,\|\eta'\|dP \geqslant \frac{1}{n!}\,\frac{n-k-1}{n-k}\sum_{\sigma\in R}\int_{\Omega}\|\eta_{\sigma(E)}\|\,\|\eta_{\sigma(S\setminus E)}\|dP$$

for $k \leqslant [n/2]$ and $E = \{1, \ldots, k\}$.

Thus, to complete the proof it is enough to show (15) and (16). The inequality (15) follows from the obvious formulas

$$(17)\qquad \frac{1}{n!}\sum_{\sigma\in R}\int_{\Omega}[\|\eta\| - \|-\eta_{\sigma(E)}+\eta_{\sigma(S\setminus E)}\|]^2dP \leqslant \frac{4}{n!}\sum_{\sigma\in R}\int_{\Omega}\|\eta_{\sigma(E)}\|\,\|\eta_{\sigma(S\setminus E)}\|dP$$

and

$$(18)\qquad \frac{1}{n!}\sum_{\sigma\in R}\int_{\Omega}[\|\eta\| - \|-\eta_{\sigma(E)}+\eta_{\sigma(S\setminus E)}\|]^2dP$$

$$= \frac{1}{n!}\sum_{\sigma\in R}\int_{\Omega}\|\eta\|^2dP + \frac{1}{n!}\sum_{\sigma\in R}\int_{\Omega}\|-\eta_{\sigma(E)}+\eta_{\sigma(S\setminus E)}\|^2dP -$$

$$-\frac{2}{n!}\sum_{\sigma\in R}\int_{\Omega}\|\eta\|\,\|-\eta_{\sigma(E)}+\eta_{\sigma(S\setminus E)}\|dP$$

$$= 2\int_{\Omega}\|\eta\|^2dP - \frac{2}{P(\Omega^k)}\int_{\Omega^k}\|\eta\|\,\|\eta'\|dP$$

by (14) and the equality

$$\int_{\Omega} \|-\eta_{\sigma(E)}+\eta_{\sigma(S\setminus E)}\|^2 dP = \int_{\Omega} \|\eta\|^2 dP$$

which holds for every $\sigma \in R$.

Now let us prove (16). First we show the following fact: if $l \leqslant k$, then

$$\frac{1}{P(\Omega^k)} \int_{\Omega^k} \|\eta\|\,\|\eta'\| dP \geqslant \frac{4}{n!} \sum_{\sigma\in R} \int_{\Omega} \|\eta_{\sigma(E)}\|\,\|\eta_{\sigma(S\setminus E)}\| dP, \tag{19}$$

where $E = \{1, \ldots, 2l\}$.

Taking the set $F = \{1, \ldots, l, 2l+1, \ldots, k+l\}$ and successively $F = \{l, \ldots, k+1\}$ in equality (14), summing the obtained equalities, and applying the triangle inequality, we get

$$\frac{1}{P(\Omega^k)} \int_{\Omega^k} \|\eta\|\,\|\eta'\| dP \geqslant \frac{1}{n!} \sum_{\sigma\in R} \int_{\Omega} \|\eta\|\,\|\eta_{\sigma(S\setminus(E\cup G))} - \eta_{\sigma(G)}\| dP, \tag{20}$$

where $G = \{k+l+1, \ldots, n\}$.

Let us note that, for an arbitrary system of signs $(\delta_1, \ldots, \delta_n)$ (i.e. $\delta_i = \pm 1$ for $i = 1, \ldots, n$), the sequence of random variables $(\varepsilon_1, \ldots, \varepsilon_n)$ has the same distribution as the sequence $(\delta_1\varepsilon_1, \ldots, \delta_n\varepsilon_n)$. In particular, given $\sigma \in R$, we can put $-\varepsilon_i$ instead of ε_i for $i \in \sigma(S\setminus E)$ in the right-hand side of inequality (20), obtaining

$$\frac{1}{P(\Omega^k)} \int_{\Omega^k} \|\eta\|\,\|\eta'\| dP \geqslant \frac{1}{n!} \sum_{\sigma\in R} \int_{\Omega} \|\eta_E - \eta_{S\setminus E}\|\,\|\eta_{\sigma(S\setminus(E\cup G))} - \eta_{\sigma(G)}\| dP. \tag{21}$$

Summing (20) and (21) and using the triangle inequality we get

$$\frac{1}{P(\Omega^k)} \int_{\Omega^k} \|\eta\|\,\|\eta'\| dP \geqslant \frac{1}{n!} \sum_{\sigma\in R} \int_{\Omega} \|\eta_E\|\,\|\eta_{\sigma(S\setminus(E\cup G))} - \eta_{\sigma(G)}\| dP. \tag{22}$$

Finally, changing ε_i for $-\varepsilon_i$ for $i \in \sigma(G)$ we obtain (19). This proves (16) an even k.

for Now, let us consider the remaining case $k = 2l-1$. Putting $-\varepsilon_{2l}$ instead of ε_{2l} in the right-hand side of (19), adding the obtained inequality to (19), and applying the triangle inequality, we get

$$\frac{1}{P(\Omega^k)} \int_{\Omega^k} \|\eta\|\,\|\eta'\| dP \geqslant \frac{1}{n!} \sum_{\sigma\in R} \int_{\Omega} \|\eta_{\sigma(F)}\|\,\|\eta_{\sigma(S\setminus E)}\| dP, \tag{23}$$

where $F = \{1, \ldots, k\}$.

Let $\tau_j \in R$ be defined for $j \in S\setminus F$ by $\tau_j(h) = h$ for $h \neq j$ and $h \neq 2l$, $\tau_j(2l) = j$, $\tau_j(j) = 2l$. Since R is a group with respect to the composition of permutations, we have

$$(24)\qquad \frac{1}{n!}\sum_{\sigma\in R}\int_{\Omega}\|\eta_{\sigma(F)}\|\,\|\eta_{\sigma(S\setminus E)}\|dP = \frac{1}{n!}\sum_{\sigma\in R}\int_{\Omega}\|\eta_{\sigma\tau_j(F)}\|\,\|\eta_{\sigma\tau_j(S\setminus E)}\|dP$$
$$= \frac{1}{n!}\sum_{\sigma\in R}\int_{\Omega}\|\eta_{\sigma(F)}\|\,\|\eta_{\sigma(S\setminus F)}-\varepsilon_{\sigma(j)}a_{\sigma(j)}\|dP.$$

Averaging the right-hand side of equality (24) over j ($j = 2l, \ldots, n$) and applying (23) we get

$$\frac{1}{P(\Omega^k)}\int_{\Omega^k}\|\eta\|\,\|\eta'\|dP$$
$$\geqslant \frac{1}{n!}\,\frac{1}{n-2l+1}\sum_{j=2l}^{n}\sum_{\sigma\in R}\int_{\Omega}\|\eta_{\sigma(F)}\|\,\|\eta_{\sigma(S\setminus F)}-\varepsilon_{\sigma(j)}a_{\sigma(j)}\|dP$$
$$= \frac{1}{n!}\,\frac{1}{n-k}\sum_{\sigma\in R}\int_{\Omega}\|\eta_{\sigma(F)}\|\sum_{j=2l}^{n}\|\eta_{\sigma(S\setminus F)}-\varepsilon_{\sigma(j)}a_{\sigma(j)}\|dP$$
$$\geqslant \frac{1}{n!}\,\frac{n-k-1}{n-k}\sum_{\sigma\in R}\int_{\Omega}\|\eta_{\sigma(F)}\|\,\|\eta_{\sigma(S\setminus F)}\|dP.$$

Thus we have proved (16), which completes the proof of Theorem 2.

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