COLLOQUIUM MATHEMATICUM

XXVI DÉDIÉ À M. WŁADYSŁAW ŚLEBODZIŃSKI 1972

EXTREMAL SUBSPACES IN AN AREAL SPACE

BY

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1. Introduction. This paper is concerned with some aspects of the geometry of spaces whose fundamental invariant is a parameter-invariant multiple integral. In a special case the integrand function is the volume element in a Riemannian space, and the geometry is determined by the metric two-index tensor giving the inner product of two tangent vectors. The difficulty of deriving a two-index metric tensor in the general case has been the subject of a series of papers by Iwamoto, Katsurada, Kawaguchi and Tandai published in Tensor in 1960-1970. Others, such as Barthel [1], Buchin Su [4] and Rund [10] have been concerned with the possibility of building up a geometrical theory of multiple integrals without a two-index metric tensor at all. One universal objective has been to find a connection which is "Euclidean" in some sense. When a two-index metric tensor exists, the Euclidean connection is expressed by the vanishing of the absolute differential of the metric tensor. When no such two-index metric tensor exists, some equivalent concept is introduced. In Barthel's treatment both the metric and the connection coefficients refer to multivectors. In Buchin Su's treatment the vanishing of the absolute derivative of the two-index metric tensor is replaced by the requirement that a set of "equations of connection" are satisfied. Another objective has been to express the Euler-Lagrange equations characterizing extremal subspaces in a simple way in terms of the connection and tensors derived on using it. Buchin Su sets this down as his second postulate for the determination of the connection, but he did not obtain the connection in terms of the fundamental integrand function using these postulates. Barthel introduced a concept of Mean Curvature, but the condition for an extremal subspace could not be expressed in terms of that notion alone.

In two recent papers, the author has returned to these questions. In one paper [7], it is proved that, using generalized Christoffel symbols introduced by Rund [10], it is possible to determine a connection which will satisfy the "equations of connection" of Buchin Su as well as his second postulate relating to the Euler-Lagrange equations. In another paper [8], it is proved that when a two-index tensor exists, it is possible to derive a Euclidean connection entirely from the integrand function, but that the connection so obtained will only satisfy the second postulate of Buchin Su on the condition for extremal subspaces provided the areal space reduces to one of the three special cases associated with the names of Finsler, Cartan, or Riemann.

We start with some of the well-known concepts necessary for dealing with Areal spaces, including the possibility of introducing a two-index metric tensor. There follows an outline of the possible treatment of an m-dimensional subspace A_m in an areal space A_n, including the first, second and third fundamental forms of A_m in A_n, with an indication of the difficulty of arriving at a suitable treatment of extremal subspaces in this approach. Finally, we dispense with the metric tensor and give the first and second variation of the multiple integral in terms of the tensors derivable from a connection satisfying the two postulates of Buchin Su.

2. Areal spaces. We consider a region of an n-dimensional differentiable manifold which is covered by one coordinate neighbourhood with coordinates x^i $(i = 1, \ldots, n)$ and a subspace of dimension $m < n$ given parametrically by

$$x^i = x^i(u^1, \ldots, u^m) = x^i(u^a) \qquad (a = 1, \ldots, m).$$

Let

$$S = \int_{(m)} L\left[x(u), \frac{\partial x}{\partial u}(u)\right] du^1 \ldots du^m \tag{2.1}$$

be a parameter-invariant integral ([10], p. 268) over a region (m) of the subspace bounded by a fixed $(m-1)$-dimensional boundary. The L is a function of $n+nm$ variables x^i and p^i_a of class C^4 at least in all the variables and satisfies:

(a) $L(x, p_a) > 0$ for linearly independent p_a;

(b) $L(x, \lambda^{a'}_a p_{a'}) = \lambda L(x, p_a)$ for $\lambda = \det \lambda^{a'}_a > 0$;

(c) writing $L;^a_i \equiv \partial^a_i L \equiv \partial L/\partial p^i_a$ and $p^a_i = \partial \log L/\partial p^i_a = L^{-1} L;^a_i$ we have the relation $L;^a_i\, p^i_\beta = \delta^a_\beta L$ or $p^a_i p^i_\beta = \delta^a_\beta$.

For later use, we also write

$$\beta^i_j = p^i_a p^a_j, \qquad \gamma^i_j = \delta^i_j - \beta^i_j. \tag{2.2}$$

The Legendre form will be

$$L^{a\beta}_{ij} = L^{-2}(LL;^{a\beta}_{ij} - L;^a_i L;^\beta_j + L;^a_j L;^\beta_i). \tag{2.3}$$

A function L satisfying the above-mentioned conditions may be expressed in terms of simple m-vectors

$$p^{i_1 \cdots i_m} = p^I = m! p^{i_1}_{[1} \cdots p^{i_m}_{m]}, \tag{2.4}$$

where we follow Barthel [1] in using a capital letter to indicate a composite index. It is well known that there exists a function $f(x, p^I)$ such that

$$f(x, p^I) = L(x^i, p^i_\alpha) \tag{2.5}$$

and studies on areal spaces have often been based on this function f rather than on the L originally given. If we write $F = L^{2/m}$, we follow Rund [10], p. 288, in introducing the tensor densities

$$g^{\alpha\beta}_{ij} = \frac{m}{2} F_{;ij}^{;\alpha\beta}. \tag{2.6}$$

We then introduce a covariant vector density of components ϱ_α such that

$$g_{ij}(x, p) = \varrho_\alpha \varrho_\beta g^{\alpha\beta}_{ij} \tag{2.7}$$

are the components of a covariant tensor of the second order. We impose the further condition on the ϱ_α that if we form

$$b_{\alpha\beta} = g_{ij} p^i_\alpha p^j_\beta, \tag{2.8}$$

then

$$\det(b_{\alpha\beta}) = L^2. \tag{2.9}$$

Since the determinant of the $b_{\alpha\beta}$ does not vanish in view of condition (2.9), we can define its reciprocal $b^{\alpha\beta}$ and hence write

$$p^\alpha_i = b^{\alpha\beta} g_{ij} p^j_\beta. \tag{2.10}$$

The tensor determined by (2.7) subject to condition (2.9) can serve as a two-index metric tensor in the areal space A_n, and it is possible to obtain connection parameters (deducible from the L and its derivatives) which will have the property that the associated covariant derivative will satisfy

$$\nabla_k g_{ij} = 0. \tag{2.11}$$

We refer for details to the author's paper [7]. This is done by considering an osculating Riemannian space along a curve determined by a relation of the form

$$dp^i_\alpha + G^i_{\alpha j}(x, p) dx^j = 0, \tag{2.12}$$

where the functions G are homogeneous of the first degree in the p variables. The three-index symbols of Christoffel for this osculating Riemannian space are given by

$$(2.13) \qquad 2\overset{\circ}{\Gamma}{}_{i}{}^{h}{}_{j} = g^{hk}(e_i g_{jk} + e_j g_{ik} - e_k g_{ij}),$$

where $e_i = \partial_i - G_{a}{}^{l}{}_{i}\partial^{a}_{l}$.

Condition (2.11) is therefore satisfied by the construction of the functions $\overset{\circ}{\Gamma}$. We have to choose the functions G to depend entirely on the function L and its partial derivatives. There are various possibilities given in [6] and in this paper the choice was $G_{a}{}^{i}{}_{j} = G_{k}{}^{i}{}_{j} p^{k}_{a}$, where

$$(2.14) \qquad G_{j}{}^{i}{}_{k} = \begin{Bmatrix} i \\ j\ k \end{Bmatrix} - C^{i}_{j};{}^{\nu}_{m} \begin{Bmatrix} m \\ n\ \ k \end{Bmatrix} p^{n}_{\nu},$$

the $\begin{Bmatrix} i \\ j\ k \end{Bmatrix}$ are the three-index symbols of Christoffel formed from the g_{ij} defined in (2.7), and where

$$(2.15) \qquad C^{i}_{j};{}^{\nu}_{m} = g^{ih} C_{jh};{}^{\nu}_{m}$$

with

$$(2.16) \qquad 2C_{ij};{}^{a}_{k} = g_{ij};{}^{a}_{k} + g_{mn};{}^{a}_{k}\gamma^{m}_{i}\beta^{n}_{j} - g_{mn};{}^{a}_{k}\gamma^{m}_{j}\beta^{n}_{i},$$

so that

$$(2.17) \qquad C^{i}_{j};{}^{a}_{k}\gamma^{n}_{i}p^{j}_{\delta} = 0.$$

The Euclidean connection equations (2.11) could therefore be expressed by

$$\frac{\partial g_{ij}}{\partial x^k} = g_{im}\Gamma_{j}{}^{m}{}_{k} + g_{mj}\Gamma_{i}{}^{m}{}_{k} = g_{ij,k},$$

$$(2.18) \qquad g_{ij};{}^{a}_{k} = g_{im}C^{m}_{j};{}^{a}_{k} + g_{mj}C^{m}_{i};{}^{a}_{k},$$

where we have put

$$(2.19) \qquad \Gamma_{j}{}^{i}{}_{k} = \overset{\circ}{\Gamma}{}_{j}{}^{i}{}_{k} + G_{a}{}^{l}{}_{j}C^{i}_{k};{}^{a}_{l},$$

and we can express the absolute differential of any vector field X in the form

$$(2.20) \qquad DX^i = dX^i + \Gamma_{j}{}^{i}{}_{k}X^j dx^k + C^{i}_{j};{}^{a}_{k}X^j dp^{k}_{a} = dX^i + \omega^{i}_{j}X^j.$$

3. Subspaces. With a subspace A_m given parametrically by

$$(3.1) \qquad x^i = x^i(u^a) \quad \text{and} \quad p^{i}_{a} = \frac{\partial x^i}{\partial u^a}$$

we consider a vector field X^i tangent to A_m, so that its components as a tangent vector field to A_m can be given in terms of the coordinate system

u^a in A_m, by ξ^a, with $X^i = p^i_a \xi^a$ and $\xi^a = p^a_i X^i$. We define an absolute differential for the ξ^a by the requirement

$$D\xi^a = d\xi^a + \omega^a_\beta \xi^\beta = p^a_i DX^i$$

which immediately gives us

$$\omega^a_\beta = p^a_i(dp^i_\beta + \omega^i_j p^j_\beta), \tag{3.2}$$

so that, for the "mixed" tensor p^i_a itself, we have

$$Dp^i_a = dp^i_a + \omega^i_j p^j_a - \omega^\beta_a p^i_\beta$$

or, assuming (3.2) and (2.2),

$$Dp^i_a = \gamma^i_j(dp^j_a + \omega^j_k p^k_a). \tag{3.3}$$

Since the functions $C^i_j;{}^a_k$ satisfy (2.17), this simplifies further to

$$Dp^i_a = \gamma^i_j(dp^j_a + \Gamma_m{}^j{}_n p^m_a dx^n). \tag{3.4}$$

From the definition of Γ given in (2.19) we can further write

$$Dp^i_a = \gamma^i_j(p_a{}^j{}_\beta + \mathring{\Gamma}_{mn} p^m_a p^n_\beta)du^\beta = \omega_a{}^i{}_\beta du^\beta. \tag{3.5}$$

If, with respect to the metric g_{ij}, we introduce the $n-m$ vectors q^i_r $(r, s = m+1, \ldots, n)$ with

$$g_{ij}q^i_r q^j_s = \delta_{rs}, \quad g_{ij}q^i_r p^j_a = 0, \quad q^r_i = g_{ij}q^i_r, \tag{3.6}$$

we can define the *second fundamental form* of A_m with respect to the r^{th} normal by

$$q^r_i D^2 x^i = q^r_i \omega_a{}^i{}_\beta du^a du^\beta = \omega_a{}^r{}_\beta du^a du^\beta. \tag{3.7}$$

The *third fundamental form* can also be introduced as follows:

Let l be the unit m-vector corresponding to the m-vector $p_{[1} \cdots p_{m]}$, where (for the measure of multivectors, we refer to Duschek and Mayer [9], p. 49) we have

$$D(p_{[1} \cdots p_{m]}) = m!\,[(Dp_{[1})p_2 \cdots p_{m]} + \cdots + p_{[1} \cdots p_{m-1}(Dp_{m]})].$$

If we denote by $d\theta$ the angle between the m-vectors l and $l+Dl$, we have

$$\cos^2 d\theta = \frac{\langle l, l+Dl\rangle}{\langle l, l\rangle\langle l+Dl, l+Dl\rangle} = \frac{1}{1+|Dl|^2},$$

so that $\sin^2 d\theta = |Dl|^2$.

Using the abbreviation $\theta_{\alpha\beta} = g_{ij}\omega_\alpha{}^i{}_\gamma \omega_\beta{}^j{}_\delta du^\gamma du^\delta$, we get

$$|Dl|^2 = \frac{1}{b}\sum_{\alpha,\beta} \begin{vmatrix} b_{11} & \cdots & b_{1m} \\ \cdots & \cdots & \cdots \\ \cdots & \theta_{\alpha\beta} & \cdots \\ b_{m1} & \cdots & b_{mm} \end{vmatrix} = b^{\alpha\beta}\theta_{\alpha\beta} \tag{3.8}$$

which, for the case of Riemannian spaces, reduces to the "angular form" appearing in Bortolotti [3], and which, in turn, is the generalization of the third fundamental form of a surface.

In terms of the metric coefficients which have been introduced in this section, the Legendre form can be written as

$$L_{ij}^{\alpha\beta} = b^{\alpha\delta} g_{mn};{}_j^{\beta} \gamma_i^m p_\delta^n + b^{\alpha\delta} g_{im} \gamma_j^m . \tag{3.9}$$

The first term is written ${}^*L_{ij}^{\alpha\beta}$ and is called the *ecmetric tensor* by Kawaguchi, who proved that the vanishing of this tensor is the necessary and sufficient condition that the general areal space reduce to one of the "model" spaces of Finsler, Cartan, or Riemann.

If we express the condition

$$E_i(L) \equiv \frac{d}{du^a}\left(L;{}_i^a\right) - L_{,i} = 0 \tag{3.10}$$

in terms of the connection parameters $\overset{\circ}{\Gamma}$ of (2.13), we obtain

$$L_{ij}^{\alpha\beta} \omega_{\alpha}{}^{i}{}_{\beta} + \beta_n^m \overset{\circ}{\Gamma}{}_m{}^n{}_k;{}_j^a\, p_a^k = 0. \tag{3.11}$$

In the general case, therefore, condition (3.11) would only simplify if connection coefficients can be found for which

$$\beta_n^m \overset{\circ}{\Gamma}{}_m{}^n{}_k;{}_j^a\, p_a^k = 0 \tag{3.12}$$

which is just the second postulate put forward by Buchin Su for the determination of a connection in areal spaces. In a recent paper [8], the author has expressed condition (3.12) in terms of the ecmetric tensor and proved that condition (3.12) will only hold for any connection coefficients $\overset{\circ}{\Gamma}$ obtained by the method of section 2 if we are dealing with one of the special cases of Finsler, Cartan, or Riemann. We may therefore state

THEOREM I. *From the integrand function L of a parameter-invariant m-fold integral there can be defined a two-index metric tensor and a connection such that the covariant derivative of the metric tensor vanishes. For an m-dimensional subspace the first, second and third fundamental forms can be defined. The condition for a minimal subspace can only be expressed in the simple form*

$$L_{ij}^{\alpha\beta} \omega_{\alpha}{}^{i}{}_{\beta} = 0$$

for the particular cases $m = 1$ or $m = n-1$ or for the case where the areal space reduces to Riemannian space.

4. First and second variations of a multiple integral. In his treatment of the Calculus of Variations for multiple integrals, Rund [10] has expressed the Euler-Lagrange equations for the vanishing of the first variation in terms of a four-index metric tensor which he has introduced. If we write $F = L^{2/m}$ and use the abbreviation

$$E_i(F) \equiv \frac{d}{du^a}(F;_j^a) - F_{,i} \tag{4.1}$$

corresponding to what has been written for L in (3.10), we have

$$E_i(F) = \frac{2}{m} L^{2/m-1} E_i(L) + \frac{2}{m}\left(\frac{2}{m} - 1\right) L^{2/m-2} \frac{dL}{du^a} L;_i^a, \tag{4.2}$$

whence, using the identity

$$E_i(L) p_a^i \equiv 0, \tag{4.3}$$

we deduce

$$E_i(F) p_a^i = \frac{2}{m}\left(\frac{2}{m} - 1\right) L^{2/m-1} \frac{dL}{du^a}. \tag{4.4}$$

Using this and (2.2), we can give the relation

$$\frac{2}{m} L^{2/m-1} E_i(L) = E_j(F) \gamma_i^j \tag{4.5}$$

which is equivalent to a relation given by Rund [10], p. 290.

The metric tensor is defined by (2.6) which is related to the Legendre form by

$$g_{ij}^{\alpha\beta} \gamma_k^j = L^{2/m} L_{ik}^{\alpha\beta}. \tag{4.6}$$

Thus we can write

$$E_k(L) = L^{(1-2/m)} g_{ij}^{\alpha\beta} \gamma_k^j \left[p_{\alpha\beta}^i + \begin{Bmatrix} i\varepsilon \\ hm\alpha \end{Bmatrix} p_\varepsilon^h p_\beta^m \right], \tag{4.7}$$

where the generalized symbols of Christoffel appear in the brackets on the right-hand side.

Using the identity

$$L;_{ij}^{\alpha\beta} p_\gamma^i = \delta_\gamma^\alpha L;_j^\beta - \delta_\gamma^\beta L;_\gamma^a \tag{4.8}$$

and expressing $F;_{ij}^{\alpha\beta}$ in terms of the corresponding expressions for L, we deduce

$$g_{ij}^{\alpha\beta} \beta_m^i \gamma_n^j = 0, \tag{4.9}$$

so that if we introduce the notation

$$\omega_\alpha{}^i{}_\beta = \gamma_k^i \left(p_\alpha{}^k{}_\beta + \begin{Bmatrix} k\varepsilon \\ hm\alpha \end{Bmatrix} p_\varepsilon^h p_\beta^m \right), \tag{4.10}$$

then the vanishing of the first variation is characterized by

(4.11) $$g^{\alpha\beta}_{ij}\,\omega_{\alpha}{}^{j}{}_{\beta} = 0.$$

The generalized symbols of Christoffel appearing on the right-hand side of (4.10) do not lend themselves to the introduction of a suitable curvature for the examination of the second variation of the multiple integral. For this purpose we use connection parameters recently introduced [7] which can be easily deduced from these generalized symbols. If we put

$$H_{\alpha}{}^{i}{}_{\beta} = \begin{Bmatrix} i\varepsilon \\ hj\alpha \end{Bmatrix} p^{h}_{\varepsilon} p^{j}_{\beta}$$

and write

(4.12) $$\Gamma_{j}{}^{i}{}_{k} = \frac{1}{m(m+1)} H_{\alpha}{}^{i}{}_{\beta};{}^{\alpha\beta}_{jk},$$

the Γ so defined satisfy the right transformation laws and also satisfy the two conditions

(4.13) $$\partial_k L - \Gamma_{j}{}^{i}{}_{k} p^{j}_{\alpha} \partial^{\alpha}_{i} L \equiv e_k L = 0$$

and

(4.14) $$\beta^{n}_{m}\,\Gamma_{n}{}^{m}{}_{k};{}^{\alpha}_{j}\, p^{k}_{\alpha} = 0$$

which were imposed by Buchin Su for the determination of connection parameters suitably related to the multiple integral. Condition (4.10) can now be written as

(4.15) $$\omega_{\alpha}{}^{i}{}_{\beta} = \gamma^{i}_{k}(p_{\alpha}{}^{k}{}_{\beta} + \Gamma_{m}{}^{k}{}_{n} p^{m}_{\alpha} p^{n}_{\beta}),$$

i.e. as a mixed tensor which is a natural generalization of the normal curvature tensor in Riemannian geometry. Using the operator e_k defined in (4.13), the curvature tensor arising from connection (4.12) is

(4.16) $$R_{i}{}^{i}{}_{kl} = e_l \Gamma_{j}{}^{i}{}_{k} + \Gamma_{m}{}^{i}{}_{l} \Gamma_{j}{}^{m}{}_{k} - k/l,$$

where k/l denotes terms obtained by interchanging the two indices k and l. We now proceed to use (4.12), (4.15) and (4.16) in obtaining an expression for the second variation of the multiple integral for the case where the first variation vanishes. If a comma denotes differentiation with respect to the coordinates x^i, and a semi-colon the corresponding differentiation with respect to p^i_α, we have $L_{,i}$ given in terms of $L;^{\alpha}_{i}$ and of the connection coefficients by (4.13). From that we obtain further

(4.17) $$L_{,i};{}^{\beta}_{j} = L;{}^{\gamma\beta}_{mj} p^{n}_{\gamma} \Gamma_{n}{}^{m}{}_{i} + L;{}^{\beta}_{m} \Gamma_{j}{}^{m}{}_{i} + L;{}^{\gamma}_{m} p^{n}_{\gamma} \Gamma_{n}{}^{m}{}_{i};{}^{\beta}_{j},$$

(4.18) $$L_{,ij} = (L;{}^{\delta\gamma}_{rm} p^{s}_{\delta} \Gamma_{s}{}^{r}{}_{j} + L;{}^{\gamma}_{r} \Gamma_{m}{}^{r}{}_{j} + L;{}^{\delta}_{r} p^{s}_{\delta} \Gamma_{s}{}^{r}{}_{j};{}^{\gamma}_{m}) p^{n}_{\gamma} \Gamma_{n}{}^{m}{}_{i} + L;{}^{\gamma}_{m} p^{n}_{\gamma} \Gamma^{m}_{ni,j}.$$

Let us now consider a 1-parameter family of subspaces of the form $x^i = x^i(u^1, \ldots, u^m; e)$ all with the same boundary of dimension $m-1$

and all reducing for $e = 0$ to $x^i = x^i(u^1, \ldots, u^m)$. Then L would be a function of e, and putting $S(e)$ for the integral corresponding to (2.1), the first and second variations of the S would be given by $S'(0)$ and $S''(0)$ for which the integrand functions would be $\partial L/\partial e|_{e=0}$ and $\partial^2 L/\partial e^2|_{e=0}$, respectively.

Putting

$$v^i = \left.\frac{\partial x^i}{\partial e}\right|_{e=0},$$

we get

$$\left.\frac{\partial L}{\partial e}\right|_{e=0} = -v^i E_i(L) = L;{}^{a}_{i} D_a v^i. \tag{4.19}$$

If we take account of the identity $E_i(L)p^i_a \equiv 0$ which is an immediate consequence of the homogeneity conditions satisfied by the functions L, we conclude that the component of v tangential to the subspace does not contribute to the integral $S'(0)$ at all, so we may assume that the variation vector v is normal to the subspace at every point. From the second equality occurring in (4.19) we can conclude

I. The first variation of S vanishes if the variation vector is parallel along the subspace.

II. The first variation of S vanishes if the covariant derivative of the variation vector is normal to the subspace at every point.

Using (4.17) and (4.18), we can obtain the integrand for the second variation when the first variation vanishes in the form

$$\left.\frac{\partial^2 L}{\partial e^2}\right|_{e=0} = L;{}^{a\beta}_{ij} D_a v^i D_\beta v^j + L\beta^s_r R^r_{isj} v^i v^j + \\ +2L\beta^s_r \Gamma_i{}^r{}_s;{}^a_j v^i D_a v^j - Lp^\beta_r v^i v^j \Gamma_i{}^r{}_j;{}^a_k \omega_a{}^k{}_\beta. \tag{4.20}$$

If the variation vector is parallel along the subspace, then the right-hand side of (4.20) simplifies and the sign of the second variation of S is determined by the tensor

$$\beta^s_r R_i{}^r{}_{sj} - p^\beta_r \Gamma_i{}^r{}_j;{}^a_k \omega_a{}^k{}_\beta. \tag{4.21}$$

When $m = n-1$, the Legendre form becomes expressible in terms of the first and second fundamental forms of the subspace as introduced in (2.8) and (3.7), and (4.20) reduces to

$$\left.\frac{\partial^2 L}{\partial e^2}\right|_{e=0} = b^{a\beta}\partial_a \omega \partial_\beta \omega - U\omega^2; \tag{4.22}$$

where ω is the variation along the unique normal at every point of the subspace. The U occurring is the invariant first introduced into the Calculus of Variations by Koschmieder, which invariant was expressed in terms of the Cartan spaces based on the notion of area by Berwald [2].

We may now state

THEOREM II. *From the integrand function L of a parameter invariant multiple integral there can be deduced a connection which is Euclidean in the sense of equation* (4.13) *on which the geometry of an areal space can be constructed. The second variation of the multiple integral for an extremal subspace may be expressed in terms of the tensors of the areal space by* (4.20). *If the variation vector is parallel along the subspace, the sign of the second variation is given in terms of the tensor* (4.21).

REFERENCES

[1] W. Barthel, *Über metrische Differentialgeometrie, begründet auf dem Begriff eines p-dimensionalen Areals,* Mathematische Annalen 137 (1959), p. 42-63.

[2] L. Berwald, *Über die n-dimensionalen Cartanschen Räume und eine Normalform der zweiten Variation eines* $(n-1)$*-fachen Oberflächintegrals,* Acta Mathematica 71 (1939), p. 191-248.

[3] E. Bortolotti, *Sulle varietà subordinate,* Rendiconti Istituto Lombardo Scientifico, (2) 64 (1931), p. 441-463.

[4] Buchin Su, *The geometry of spaces with areal metrics,* Mathematische Nachrichten 16 (1957), p. 281-287.

[5] — *On the theory of affine connections in an areal space,* Bulletin Mathématique de la Société des Mathématiques et Physiques de la République Populaire Roumaine 2 (50) (1958), p. 185-190.

[6] E. T. Davies, *Areal spaces,* Annali di Matematica Pura ed Applicata 55 (1961), p. 63-76.

[7] — Aequationes Mathematicae 7 (1972), p. 173-181.

[8] — Tensor (to appear).

[9] A. Duschek und W. Mayer, *Lehrbuch der Differentialgeometrie,* Teubner 1930.

[10] H. Rund, *The Hamilton-Jacobi theory in the calculus of variations,* London and New York 1966.

[11] — *A geometrical theory of multiple integral problems in the calculus of variations,* Canadian Journal of Mathematics 20 (1968), p. 639-657.

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Reçu par la Rédaction le 10. 12. 1971