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CLOSED SURFACES IN FOUR-SPACES WITH NON-VANISHING NORMAL CURVATURE

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We investigate surfaces immersed in 4-spaces of constant curvature by making use of the invariance of the normal curvature ellipse. The main global results concern minimal immersions with restriction to the Gauss curvature and nowhere vanishing normal curvature. An example is the Veronese surface in S^4. Furthermore, this is an example for an "isotropic" surface, which means that the normal curvature ellipse is a circle. We give sufficient conditions for closed minimal surfaces to be isotropic.

We assume that all manifolds and maps considered in this paper are C^∞. Let $f\colon M \to \bar{M}(c)$ be an isometric immersion of a connected orientable Riemannian 2-manifold (M, g) in a 4-dimensional oriented Riemannian manifold $(\bar{M}(c), \bar{g})$ with constant curvature c. Let TM (respectively, $\perp_f M$) denote the tangential (respectively, normal) bundle. D is the covariant derivative in $\perp_f M$, induced by the covariant derivative in $T\bar{M}$. Choose the orientation of $\perp_f M$ so that the canonical isomorphism $TM \oplus \perp_f M = T\bar{M}$ preserves orientation. $\alpha\colon TM \times TM \to \perp_f M$ is the second fundamental form. Let (e_1, e_2, e_3, e_4) denote a local adapted orthonormal frame field, i.e. e_1, e_2 are tangential and e_3, e_4 are normal vector fields.

The normal connection form ω satisfies

$$De_3 = \omega e_4 \quad \text{and} \quad De_4 = -\omega e_3.$$

The normal curvature tensor R^n is defined by

$$R^n(X, Y)e_3 = d\omega(X, Y)e_4, \qquad X, Y \in \Gamma(TM).$$

The normal curvature form

$$d\omega = \bar{g}\big(R\ \ (\,,\,)e_3, e_4\big)$$

is independent of the choice of e_3, e_4. The normal curvature K_n is defined by the formula

$$d\omega + K_n \omega_v = 0,$$

where ω_v is the volume element of M. This gives immediately $\int K_n \omega_v = 0$ for closed orientable surfaces with a global nowhere vanishing normal section.

For the further discussion the local normal vector fields

$$\Phi = \tfrac{1}{2}\big(a(e_1, e_1) - a(e_2, e_2)\big) \quad \text{and} \quad \Psi = a(e_1, e_2)$$

are important. The curvature ellipse is given by $\Phi \cos\varrho + \Psi \sin\varrho$.

A point $p \in M$ is called *isotropic* if $\|\Phi\| = \|\Psi\|$ and $\bar{g}(\Phi, \Psi) = 0$. We say that M is *isotropic* if each point is isotropic. At non-isotropic points, Φ and Ψ determine a tangential cross field. Therefore, on any compact surface of non-vanishing Euler characteristic there must be an isotropic point (see [4]).

$K_n \neq 0$ iff Φ and Ψ are linear independent. This follows from

LEMMA 1. *The volume elements ω_v of M and $\bar{\omega}_v$ of $\bar{M}$ satisfy*

$$K_n \omega_v(X, Y) = 2\bar{\omega}_v(X, Y, \Phi, \Psi).$$

Proof. We have

$$K_n \omega_v(X, Y) = -\bar{g}\big(R^n(X, Y)e_3, e_4\big) = \sum_{i=1,2} \big[\bar{g}\big(a(\cdot, e_i), e_3\big), \bar{g}\big(a(\cdot, e_i), e_4\big)\big](X, Y)$$

(see, e.g., [5]). For $X = e_1$ and $Y = e_2$ we get

$$K_n = \bar{g}(\Psi, e_4)\bar{g}(2\Phi, e_3) - \bar{g}(\Psi, e_3)\bar{g}(2\Phi, e_4) = 2\omega_3 \wedge \omega_4(\Phi, \Psi)$$

with $\omega_i = \bar{g}(e_i, \cdot)$ and, therefore,

$$K_n \omega_v(X, Y) = 2\omega_v(X, Y)\omega_3 \wedge \omega_4(\Phi, \Psi) = 2\bar{\omega}_v(X, Y, \Phi, \Psi).$$

We also write $K_n = 2\Phi \wedge \Psi$.

LEMMA 2. *Let M be a closed orientable surface isometrically immersed in $\bar{M}$. If $K_n \neq 0$, then $\int K_n \omega_v = 8\pi \operatorname{sign}(K_n)$ and M has genus* 0.

Proof. We assume the tangential frame e_1, e_2 to have isolated singularities at $p_1, \ldots, p_n \in M$ only. $K_n \neq 0$ implies $\Phi \neq 0$ so that we can choose $e_3 = \Phi/\|\Phi\|$. This gives the unique frame field (e_1, e_2, e_3, e_4) defined on $M \setminus \{p_1, \ldots, p_n\}$. Let U_i be a sufficiently small disk in M with center p_i. Then

$$\int_{M \setminus \cup U_i} K_n \omega_v = -\int d\omega = \sum_{i=1}^{n} \int_{\partial U_i} \omega.$$

We evaluate $\int_{\partial U_i} \omega$ suppressing the index i for this local calculation. We cover U by a continuous positively-oriented reference frame field

$\bar{e}_1, \bar{e}_2$. Let φ denote the angle from $\bar{e}_1$ to e_1 on ∂U. We have

$$e_1 = \bar{e}_1 \cos\varphi + \bar{e}_2 \sin\varphi, \qquad e_2 = -\bar{e}_1 \sin\varphi + \bar{e}_2 \cos\varphi,$$

which implies

$$\Phi = \bar{\Phi}\cos 2\varphi + \bar{\Psi}\sin 2\varphi, \qquad \Psi = -\bar{\Phi}\sin 2\varphi + \bar{\Psi}\cos 2\varphi$$

if we set $\bar{\Phi} = \frac{1}{2}\big(\alpha(\bar{e}_1, \bar{e}_1) - \alpha(\bar{e}_2, \bar{e}_2)\big)$ and $\bar{\Psi} = \alpha(\bar{e}_1, \bar{e}_2)$. Let ϱ be the angle from $\bar{\Phi}$ to Φ. We have

$$\operatorname{sign}(K_n) \int_{\partial U} d\varrho = 2 \int_{\partial U} d\varphi .$$

This is clearly true with respect to a normal metric g_0 for which $\bar{\Phi}$ and $\bar{\Psi}$ are orthonormal vectors. The sign on the left depends on the orientation of $\bar{\Phi}$, $\bar{\Psi}$ as seen in Lemma 1.

We set $g_t = (1-t)g_0 + tg_1$, where $g_1 = \bar{g} | \perp_f M$ is the normal metric, and denote by ϱ_t the angle with respect to the metric g_t. Then $\int_{\partial U} d\varrho_t$ becomes a continuous function of t. Indeed, this function is constant since the possible values of the integral are discrete.

Now choose $\bar{e}_3 = \bar{\Phi}/\|\bar{\Phi}\|$. Then in U we have the unique frame field $(\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4)$ and the connection form $\overline{\omega} = \bar{g}(D\bar{e}_3, \bar{e}_4)$. Since $\omega = \overline{\omega} + d\varrho$, we have

$$\int_{\partial U} \omega = \int_{\partial U} \overline{\omega} + 2 \operatorname{sign}(K_n) \int_{\partial U} d\varphi .$$

This completes the local calculation.

Poincaré's theorem (see, e.g., [6], p. 236)

$$\sum_{i=1}^{n} \int_{\partial U_i} d\varphi_i = 4\pi(1-g)$$

now implies the lemma.

The proof of Lemma 2 shows that the Euler characteristics of tangent and normal bundle are related by

$$\chi(N) = 2 \operatorname{sign}(K_n) \chi(M).$$

Little defines in [4] a symmetric tensor field S by

$$\bar{g}(dX, e_3) \wedge \bar{g}(dY, e_4) = S(X, Y)\, \omega_1 \wedge \omega_2 .$$

The symmetric tensor field L with $L_{ik} = \varepsilon_i^l S_{lk} - S_{kl} \varepsilon_i^l$ has the properties

$$\operatorname{tr} L = 0, \qquad \det(L) = 0 \text{ implies } L = 0,$$

and thus defines a tangential cross field by $L_{ik} du^i du^k = 0$.

In the generic case the singularities of L are isolated and are the same as those of the mean curvature vector H. Little compares the signs of index (L) and index (H). There is a sign error in his calculations. So he gets the equations on the Euler characteristics with the opposite sign. Besides, his method fails on minimal surfaces. As an example we have the Veronese surface $S^2(\sqrt{3}) \to S^4(1)$ with $K_n = \frac{2}{3}$. The immersion is minimal with area 12π. We know from Calabi [2] that there is no minimal immersion with less area. In case $K_n > 0$ and Gauss curvature $K \geqslant -\frac{1}{2}K_n$ we always have area 12π. To prove this we introduce the non-negative function $h\colon M \to \boldsymbol{R}$ defined by

$$h = c + H^2 - K - K_n.$$

The function h may be written as $h = \|\Phi\|^2 + \|\Psi\|^2 - 2\Phi \wedge \Psi$. This shows that the zeros of h are just the isotropic points of M. Let ω_{12} and ω_{34} denote the tangential and normal connection forms, respectively.

LEMMA 3. *Let M be a closed orientable surface isometrically and minimally immersed in $\bar{M}$ with $K_n > 0$. Then $\Delta h = Ah$, where*

$$A = 4K + 2K_n + \|4\omega_{12} + 2\omega_{34}\|^2.$$

Proof. We confine the calculations to a non-isotropic neighbourhood of a non-isotropic point. The Codazzi equations $(\nabla_X a)(Y, Z) = (\nabla_Y a)(X, Z)$ with $X = e_1$, $Y = Z = e_2$ and $X = e_2$, $Y = Z = e_1$ yield, in case of a minimal immersion,

$$-\nabla_{e_1}(\Phi) + 2\omega_{12}(e_1)\Psi = \nabla_{e_2}(\Psi) + 2\omega_{12}(e_2)\Phi \tag{1}$$

and

$$\nabla_{e_2}(\Phi) - 2\omega_{12}(e_2)\Psi = \nabla_{e_1}(\Psi) + 2\omega_{12}(e_1)\Phi, \tag{2}$$

respectively. We choose e_1, e_2 so that $\bar{g}(\Phi, \Psi) = 0$ (this is always possible and, for example, the case where Φ is the major axis of the curvature ellipse), and then multiply (1) and (2) by Φ and Ψ, respectively. We get the following equations:

$$-\tfrac{1}{2}e_1(\|\Phi\|^2) = \bar{g}(\Phi, \nabla_{e_2}\Psi) + 2\omega_{12}(e_2)\|\Phi\|^2, \tag{3}$$

$$-\bar{g}(\Psi, \nabla_{e_1}\Phi) + 2\omega_{12}(e_1)\|\Psi\|^2 = \tfrac{1}{2}e_2(\|\Psi\|^2), \tag{4}$$

$$\tfrac{1}{2}e_2(\|\Phi\|^2) = \bar{g}(\Phi, \nabla_{e_1}\Psi) + 2\omega_{12}(e_1)\|\Phi\|^2, \tag{5}$$

$$\bar{g}(\Psi, \nabla_{e_2}\Phi) - 2\omega_{12}(e_2)\|\Psi\|^2 = \tfrac{1}{2}e_1(\|\Psi\|^2). \tag{6}$$

Evaluations $(3)\cdot\|\Psi\|^2 + (6)\cdot\|\Phi\|^2$ and $(4)\cdot\|\Phi\|^2 + (5)\cdot\|\Psi\|^2$ give

$$8\omega_{12}(e_2)K_n^2 + e_1(K_n^2) = 4K_n(c-K)\,\omega_{34}(e_2),$$

$$8\omega_{12}(e_1)K_n^2 - e_2(K_n^2) = 4K_n(c-K)\,\omega_{34}(e_1)$$

or, for positively-oriented orthonormal vector fields X, $X^{\perp}$ and since $K_n \neq 0$, we obtain

$$X(K_n)+4K_n\omega_{12}(X^{\perp})-2(c-K)\,\omega_{34}(X^{\perp}) = 0. \tag{7}$$

In the same way, (4), (5) and (3), (6) yield

$$X(c-K)+4(c-K)\,\omega_{12}(X^{\perp})-2K_n\omega_{34}(X^{\perp}) = 0. \tag{8}$$

Computation of the Laplacian of K_n and $(c-K)$ by a further differentiation gives

$$\Delta K_n+4dK_n\wedge\omega_{12}(X, X^{\perp})-6KK_n+2dK\wedge\omega_{34}(X, X^{\perp})+2K_n = 0 \tag{9}$$

and

$$-\Delta K-4dK\wedge\omega_{12}(X, X^{\perp})-4K(c-K)-2dK_n\wedge\omega_{34}(X, X^{\perp})+ +2K_n^2 = 0, \tag{10}$$

where $4dK\wedge\omega_{12}+2dK_n\wedge\omega_{34} = \|\nabla a\|^2\omega_v$ (see [7]). This, together with

$$\begin{aligned}(4dK\wedge\omega_{12}+2dK_n\wedge\omega_{34})(e_1, e_2)\\ = 4(c-K)(2\|\omega_{12}\|-\|\omega_{34}\|)^2+16(c-K)\|\omega_{12}\|\,\|\omega_{34}\|-16K_n g(\omega_{12}, \omega_{34})\end{aligned}$$

and

$$\begin{aligned}(4dK_n\wedge\omega_{12}+2dK\wedge\omega_{34})(e_1, e_2)\\ = 16(c-K)g(\omega_{12}, \omega_{34})-4K_n(2\|\omega_{12}\|-\|\omega_{34}\|)^2\ -16K_n\|\omega_{12}\|\,\|\omega_{34}\|,\end{aligned}$$

gives $\Delta h = Ah$.

THEOREM 1. *Let $f\colon M\to S^4(1)$ be a minimal isometric immersion of a closed connected orientable 2-dimensional Riemannian manifold with normal curvature $K_n>0$ and Gauss curvature $K\geqslant -\frac{1}{2}K_n$. Then the immersion is isotropic and M has area 12π.*

Proof. Since $A\geqslant 0$ on M, we have $h = \text{const}$ by a lemma of E. Hopf. Indeed, $h = 0$ since A must be somewhere positive on M. We now integrate $1 = K+K_n$ and apply Lemma 2.

If M is complete and $\bar{M} = S^n(1)$, $n>2$, we know from [1] that if the immersion is minimal and $K\geqslant\frac{1}{3}$, then $f(M)$ is either totally geodesic or $f(M)$ is the Veronese surface in $S^4(1)$. Applying Theorem 1 we get

THEOREM 2. *Let $f\colon M\to S^4(1)$ be a minimal isometric immersion of a closed connected 2-dimensional Riemannian manifold with normal curvature $K_n>0$. If $K\geqslant\frac{1}{3}$ or $-\frac{1}{2}K_n\leqslant K\leqslant\frac{1}{3}$, then $K = \frac{1}{3}$, $K_n = \frac{2}{3}$, and $f(M)$ is the Veronese surface.*

Proof. According to Theorem 1 we have

$$4\pi = \int K\omega_v \geqslant (\leqslant)\, \tfrac{1}{3}\int\omega_v = 4\pi,$$

whence $K = \frac{1}{3}$ and $K_n = \frac{2}{3}$ in which case the immersed manifold is the Veronese surface (see [1]).

Remark. If $K_n > 0$ is a constant, then $K \geqslant \frac{1}{3}$ and, therefore, $f(M)$ is the Veronese surface. This result is due to Itoh [3].

Proof. By (9) we have $-6KK_n + 2dK \wedge \omega_{34}(X, X^{\perp}) + 2K_n = 0$ and from (7) we obtain $2K_n\omega_{12} = (1-K)\omega_{34}$, so that

$$dK \wedge \omega_{34} = \frac{2K_n}{1-K} dK \wedge \omega_{12} = \frac{K_n}{2(1-K)} \|\nabla a\|^2 \omega_v$$

and

$$-6KK_n + \frac{K_n}{1-K} \|\nabla a\|^2 + 2K_n = 0.$$

This gives a quadratic equation for K with the solution

$$K = \tfrac{1}{3}\left(2 \pm \sqrt{1 - \tfrac{3}{2}\|\nabla a\|^2}\right) \geqslant \tfrac{1}{3}.$$

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