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ON LOCAL ISOMETRIES AND ISOMETRIES IN METRIC SPACES

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As a special instance of results of Edelstein [2], it is known that if (M, d) is a compact and connected metric space and if $T: (M, d) \to (M, d)$ is *locally contractive* in the sense that each point of M has a neighborhood U for which $d(Tu, Tv) < d(u, v)$ whenever u and v are distinct points of U, then T has a unique fixed point. Using a remetrization argument, Rosenholtz [8] has shown that this result is a consequence of the simple fact that globally contractive self-maps of compact metric spaces have fixed points. In the sequel we show (by a different remetrization approach) that an even more general result follows quickly from the fact that surjective non-expansive mappings of compact metric spaces must necessarily be isometries. This fact was established in 1936 by Freudenthal and Hurewicz [4], who noted at the same time that expansions of compact metric spaces must also be isometries. As a companion result we include an application of this second fact to obtain a generalization of the first.

We begin by proving a result more general than needed for the fixed point theorem.

A mapping T of a metric space M into itself is said to be *locally uniformly β-lipschitzian* for fixed $\beta \geqslant 1$ if each $x \in M$ has a neighborhood U such that, for all $u, v \in U$,

$$d(T^n u, T^n v) \leqslant \beta d(u, v), \quad n = 1, 2, \ldots$$

THEOREM 1. *Let M be a compact metric space and suppose $T: M \to M$ is surjective and locally uniformly β-lipschitzian. Then T is a homeomorphism and T^{-1} is also locally uniformly β-lipschitzian on M.*

A mapping $T: M \to M$ is called *locally non-expansive* if each point $x \in M$ has a neighborhood U for which $d(Tu, Tv) \leqslant d(u, v)$ for all $u, v \in U$. For compact M, a locally non-expansive mapping is locally uniformly 1-lipschitzian, so an application of Theorem 1 with $\beta = 1$ yields the following localized version of the Freudenthal and Hurewicz result (cf. [3]).

COROLLARY 1. *Let M be a compact metric space and $T: M \to M$ surjective and locally non-expansive. Then T is a homeomorphism and a local isometry.*

Proof of Theorem 1. Since T is locally uniformly β-lipschitzian and M is compact, there exists a positive number δ such that, whenever $d(x, y) < \delta$, $d(T^n x, T^n y) \leqslant \beta d(x, y)$ for $n = 1, 2, \ldots$ Introduce a new metric r on M by defining

$$r(x, y) = \sup\{d(T^i x, T^i y): i = 0, 1, \ldots\}.$$

Locally we have

$$d(x, y) \leqslant r(x, y) \leqslant \beta d(x, y), \tag{1}$$

and thus the metrics r and d are equivalent, and the space (M, r) is also compact. Since T is non-expansive relative to the metric r, by the result of Freudenthal and Hurewicz we get

$$r(Tx, Ty) = r(x, y), \qquad x, y \in M. \tag{2}$$

With (1) this yields immediately the fact that T is a homeomorphism relative to d. Also, by (1) and (2),

$$d(x, y) \leqslant r(x, y) = r(Tx, Ty) \leqslant \beta d(x, y)$$

whenever $d(x, y) < \delta$, and since (2) and the left-hand side of (1) hold for all $x, y \in M$, this yields

$$d(T^{-1}x, T^{-1}y) \leqslant r(x, y) \leqslant \beta d(x, y), \qquad d(x, y) < \delta.$$

As a historical comment in connection with Corollary 1, we note that Edrei [3] obtained a number of results for mappings $T: M \to M$ which are locally non-expansive in the sense that each point $x \in M$ has a neighborhood U such that, for each $y \in U$, $d(Tx, Ty) \leqslant d(x, y)$. He offered the conjecture, subsequently answered in the negative by Williams [9], that a surjective mapping of a compact metric space — which is locally non-expansive in the above-defined sense — must necessarily be a local isometry. The mapping of Corollary 1 satisfies a stronger assumption of local non-expansiveness.

As an application of Corollary 1 we have

THEOREM 2. *Let M be a compact and connected metric space and suppose that $T: M \to M$ is locally non-expansive. Suppose also that T is locally contractive at each $x \in M$ in the sense that each neighborhood of x contains a point y such that $d(T^n x, T^n y) < d(x, y)$ for some positive integer n. Then T has a unique fixed point in M.*

COROLLARY 2 (see [2] and [8]). *A locally contractive mapping of a compact and connected metric space into itself has a unique fixed point.*

Proof of Theorem 2. Let $M_1 = M$ and $M_{n+1} = T(M_n)$ for $n = 1, 2, \ldots$ By continuity of T, $\{M_i\}$ is a nested sequence of non-empty compact connected subsets of M. It follows that the space

$$M_\infty = \bigcap_{n=1}^{\infty} M_n$$

is non-empty, compact, and connected and, moreover, $T(M_\infty) = M_\infty$. Hence, by Corollary 1, T is a local isometry on M_∞. Since M_∞ is connected, it follows easily from our locally contractive assumption that M_∞ must contain only a single point, which is the unique fixed point of T.

We might also remark that another consequence of Corollary 1 is the following result, related to work of Busemann [1] on length-preserving maps. Suppose that (M, d) is rectifiably pathwise connected and let l denote the intrinsic metric of M. (Thus $l(u, v) = \inf l(a)$, where $l(a)$ denotes the length of the rectifiable path a joining u and v, and the infimum is taken over all such paths.)

Corollary 3. *Let (M, d) be a compact metric space which is rectifiably pathwise connected, and suppose that $T\colon (M, d) \to (M, d)$ is surjective and locally non-expansive. Then $T\colon (M, l) \to (M, l)$ is an isometry (where l denotes the intrinsic metric of M).*

Proof. By Corollary 1, T is an injective local isometry on M; hence both T and T^{-1} are length preserving, and thus non-expansive in the metric l. Consequently, T is an isometry in l. (It is a simple exercise to show that T need not be an isometry in the metric d.)

We conclude with a generalized version of the fact that surjective non-expansive mappings on compact metric spaces are isometries. Condition (*) in the sequel, introduced by Kirk in [6], is much weaker than non-expansiveness and, in fact, does not even imply continuity of T. This theorem generalizes Theorem 3.2 of [5], and it is applied in [7] to the study of asymptotically non-expansive semigroups. The proof given here is rather detailed; a quicker proof along the lines of Theorem 1 is possible if it is assumed at the outset that T is continuous.

Theorem 3. *Let M be a compact metric space and suppose $T\colon M \to M$ is surjective. If, for each $x \in M$,*

$$(*) \qquad \limsup_{i\to\infty} \{\sup_{y\in M} [d(T^i x, T^i y) - d(x, y)]\} \leqslant 0,$$

then T is an isometry.

Proof. Fix x_0, y_0 in M and define $\{x_n\}, \{y_n\}$ by $T(x_{n+1}) = x_n$, $T(y_{n+1}) = y_n$ for $n = 0, 1, \ldots$ Since M is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ for which the corresponding subsequence $\{y_{n_k}\}$ of $\{y_n\}$ also converges. Assume that $x_{n_k} \to x_\infty$ and $y_{n_k} \to y_\infty$ as $k \to \infty$. Then, given $\varepsilon > 0$, there exists a positive integer N_1 such that, for $k \geqslant N_1$,

$$d(x_{n_k}, x_\infty) < \varepsilon \quad \text{and} \quad d(y_{n_k}, y_\infty) < \varepsilon.$$

With $\varepsilon > 0$ thus given, by (*) there exists an N_2 such that $i \geqslant N_2$ implies

$$\sup_{y \in M} [d(T^i x_\infty, T^i y) - d(x_\infty, y)] < \varepsilon,$$

i.e., $d(T^i x_\infty, T^i y) < d(x_\infty, y) + \varepsilon$ for all $y \in M$. Similarly, by (*) there exists an N_3 such that $i \geqslant N_3$ implies $d(T^i y_\infty, T^i y) < d(y_\infty, y) + \varepsilon$ for all $y \in M$. Let $N_4 = \max(N_1, N_2, N_3)$. Then if $i, k \geqslant N_4$, we get

$$d(T^i x_\infty, T^i x_{n_k}) < d(x_\infty, x_{n_k}) + \varepsilon < 2\varepsilon$$

and

$$d(T^i y_\infty, T^i y_{n_k}) < d(y_\infty, y_{n_k}) + \varepsilon < 2\varepsilon.$$

Fix $k \geqslant N_4$ and let $p_j = n_{k+j} - n_k$ for $j = 1, 2, \ldots$ Then $p_1 < p_2 < \ldots$, and $p_j \to \infty$ as $j \to \infty$. Now

$$\begin{aligned} d(x_0, T^{p_j} x_0) &= d\big(T^{n_k+p_j}(x_{n_k+p_j}), T^{n_k+p_j}(x_{n_k})\big) \\ &\leqslant d\big(T^{n_k+p_j}(x_{n_k+p_j}), T^{n_k+p_j}(x_\infty)\big) + d\big(T^{n_k+p_j}(x_\infty), T^{n_k+p_j}(x_{n_k})\big) \\ &< d(x_{n_k+p_j}, x_\infty) + \varepsilon + d(x_\infty, x_{n_k}) + \varepsilon < 4\varepsilon. \end{aligned}$$

Similarly, $d(y_0, T^{p_j} y_0) < 4\varepsilon$ for $j = 1, 2, \ldots$ By (*) again, for $\varepsilon > 0$ there exists an $N_5 > 0$ such that if $i \geqslant N_5$, then

$$\sup_{y \in M} \big[d\big(T^i(Tx_0), T^i(y)\big) - d(Tx_0, y)\big] < \varepsilon.$$

Thus for $y \in M$ and $i \geqslant N_5$ we get

$$d\big(T^i(Tx_0), T^i(y)\big) < d(Tx_0, y) + \varepsilon$$

and, in particular,

$$d\big(T^{i+1}(x_0), T^{i+1}(y_0)\big) < d(Tx_0, Ty_0) + \varepsilon.$$

Since $p_j \to \infty$ as $j \to \infty$, it is possible to choose p_j such that $p_j - 1 \geqslant N_5$, whence

$$\begin{aligned} d(Tx_0, Ty_0) + \varepsilon &> d(T^{p_j} x_0, T^{p_j} y_0) \\ &\geqslant d(x_0, y_0) - d(x_0, T^{p_j} x_0) - d(T^{p_j} y_0, y_0) \\ &> d(x_0, y_0) - 4\varepsilon - 4\varepsilon. \end{aligned}$$

Since ε is arbitrary, we have $d(Tx_0, Ty_0) \geqslant d(x_0, y_0)$, proving that T is expansive on M. It now follows from the expansion result of Freudenthal and Hurewicz [4] that T is an isometry.

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