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RELATIONS AMONG SOME CLOSED GRAPH AND OPEN MAPPING THEOREMS

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1. Introduction. In the recent paper [5], Byczkowski and Pol proved some closed graph and open mapping theorems for Čech-complete topological spaces. Their results imply the corresponding ones of Banach [1] and [2], Klee [11], Weston [19], Brown [3], and Pettis [12]-[14]. They do not imply some sharper results known for completely metrizable topological vector spaces (Pták [16] and [17], A. Robertson and W. Robertson [18]), locally compact topological groups (Kelley [10]), completely metrizable topological groups (Kelley [10], Husain [9]), and Čech-complete topological groups (Brown [4]). All theorems under consideration use the notions of nearly continuity and nearly openness and appear to be superficially similar. However, their proofs involve various techniques and sometimes apply deep incidental results.

In this paper we present a unified approach to the subject. It is based on the notion of Δ-closed graph, which is stronger than that of closed graph and weaker than continuity (cf. Section 2). We use a construction due to Weston [19] and Byczkowski and Pol [5] to prove a Δ-closed graph theorem for Čech-complete topological spaces (Theorem in Section 3). Then we derive from it the most general of the mentioned results (cf. Section 4). Our closed graph (Corollary 4) and open mapping (Corollary 7) theorems for Čech-complete topological groups improve previous theorems of this kind.

2. Functions with Δ-closed graph. Let E be a topological space. We start with two propositions concerning a convergence-type property of nets in the product space $E\times E$. The diagonal is denoted by Δ_E. A set has diameter less than an open cover $\mathscr{C}$ if it is in some member of $\mathscr{C}$.

PROPOSITION 1. *Let $\{(a_\sigma, b_\sigma)\}$ be a net in $E\times E$. The following conditions are equivalent*:

(i) *for any open set $W\supset \Delta_E$ the points (a_σ, b_σ) are eventually in W (i.e., there is σ_0 such that $(a_\sigma, b_\sigma)\in W$ for all $\sigma\geqslant\sigma_0$)*;

(ii) *for any open cover $\mathscr{C}$ of E the sets $\{a_\sigma, b_\sigma\}$ have eventually diameter less than $\mathscr{C}$.*

Proof. (i) $\Rightarrow$ (ii). Given $\mathscr{C}$, apply (i) to

$$W = \bigcup_{U \in \mathscr{C}} U \times U.$$

(ii) $\Rightarrow$ (i). Given W, there exists a $\mathscr{C}$ such that

$$\bigcup_{U \in \mathscr{C}} U \times U \subset W;$$

apply (ii) to $\mathscr{C}$.

If the conditions of Proposition 1 hold, we will write $(a_\sigma, b_\sigma) \to \Delta_E$.

In case E is a regular ($= T_3$) space, $(a_\sigma, b_\sigma) \to \Delta_E$ and $(a_\sigma, b_\sigma) \to (a, b)$ imply $(a, b) \in \Delta_E$. It turns out that the validity of this implication is characteristic for Urysohn spaces. E is said to be a *Urysohn space* if any two distinct points $a, b \in E$ can be separated with open sets $U_1, U_2 \subset E$ whose closures are disjoint (cf. Engelking [6], Problem 1.7.7).

PROPOSITION 2. *The following are equivalent:*

(i) *E is a Urysohn space;*

(ii) *for any point $(a, b) \in (E \times E) \setminus \Delta_E$ there exist disjoint open sets $W_1, W_2 \subset E \times E$ such that $(a, b) \in W_1$ and $\Delta_E \subset W_2$;*

(iii) *$(a_\sigma, b_\sigma) \to \Delta_E$ and $(a_\sigma, b_\sigma) \to (a, b)$ imply $(a, b) \in \Delta_E$.*

Proof. (i) $\Rightarrow$ (ii). Put $W_1 = U_1 \times U_2$ and $W_2 = (E \times E) \setminus \overline{W}_1$; $\Delta_E \subset W_2$ because $\Delta_E \cap \overline{W}_1 = \Delta_E \cap (\overline{U}_1 \times \overline{U}_2) = 0$.

(ii) $\Rightarrow$ (i). There are open sets $U_i \subset E$ such that $(a, b) \in U_1 \times U_2 \subset W_1$; since $(\overline{U}_1 \times \overline{U}_2) \cap \Delta_E \subset \overline{W}_1 \cap W_2 = 0$, $\overline{U}_1 \cap \overline{U}_2 = 0$.

The equivalence of (ii) and (iii) can be proved analogously to the characterization of Hausdorff spaces as those in which each net has at most one limit (cf. Engelking [6], Proposition 1.6.7).

Let E, F be topological spaces, and let f be a function on E to F.

Definition. The function f has a *Δ-closed graph* if $(a_\sigma, b_\sigma) \to \Delta_E$ and $(f(a_\sigma), f(b_\sigma)) \to (c, d)$ imply $(c, d) \in \Delta_F$.

PROPOSITION 3. *If f has a Δ-closed graph, then f has a closed graph.*

Proof. $a_\sigma \to a$ and $f(a_\sigma) \to c$ imply $(c, f(a)) \in \Delta_F$.

PROPOSITION 4. *Suppose F is a Urysohn space. If f is continuous, then f has a Δ-closed graph.*

Proof. Given an open cover $\mathscr{C}$ of F, $f^{-1}(\mathscr{C})$ is an open cover of E. Hence, in view of Proposition 1, $(a_\sigma, b_\sigma) \to \Delta_E$ implies $(f(a_\sigma), f(b_\sigma)) \to \Delta_F$, which yields the assertion (see Proposition 2, (i) $\Rightarrow$ (iii)).

There are two important classes of — not necessarily continuous — functions which have Δ-closed graph. They will be pointed out in the successive two propositions.

A function f is called *inversely subcontinuous* if any net $\{a_\sigma\} \subset E$ has a convergent subnet provided $\{f(a_\sigma)\}$ is convergent (cf. Fuller [8]). This is a generalization of a function with a compact domain and of a one-to-one function onto ($=$ bijection) with a continuous inverse.

LEMMA. *Suppose that*

(1) *f is inversely subcontinuous and has a closed graph.*

Then for any net $\{(a_\sigma, b_\sigma)\} \subset E \times E$, which has no cluster points out of Δ_E, $(f(a_\sigma), f(b_\sigma)) \to (c, d)$ implies $(c, d) \in \Delta_F$.

Proof. Applying twice the inverse subcontinuity of f we get a subnet of $\{(a_\sigma, b_\sigma)\}$ which converges to a point $(a, b) \in E \times E$. By the assumption on cluster points, $(a, b) \in \Delta_E$. Since f has a closed graph, $c = f(a)$ and $d = f(b)$. Hence $c = d$.

PROPOSITION 5. *Suppose E is a Urysohn space. If f satisfies* (1), *then f has a Δ-closed graph.*

Proof. If $(a_\sigma, b_\sigma) \to \Delta_E$, the same property has any of its subnets; by Proposition 2, (i) $\Rightarrow$ (iii), the net has no cluster points out of Δ_E. The assertion follows from the Lemma.

PROPOSITION 6. *Suppose that E, F are topological groups and f is a homomorphism. Then f has a closed graph (if and) only if it has a Δ-closed graph.*

Proof. $(a_\sigma, b_\sigma) \to \Delta_E$ yields $a_\sigma b_\sigma^{-1} \to 1_E$, and $(f(a_\sigma), f(b_\sigma)) \to (c, d)$ yields $f(a_\sigma b_\sigma^{-1}) \to cd^{-1}$; hence $cd^{-1} = f(1_E) = 1_F$ (provided f has a closed graph).

Now we will examine condition (1). Recall that a function is said to be *closed* if the image of any closed set is closed. A Tychonoff space F is *Čech-complete* if it is a dense G_δ in a compact Hausdorff ($= T_2$) space; locally compact spaces and completely metrizable spaces are Čech-complete (cf. Engelking [6], Section 3.9, and Theorem 4.3.26). A Hausdorff space F is a *k-space* if it is an image of a locally compact space under a quotient mapping; this is so if F is first-countable or Čech-complete (cf. Engelking [6], Section 3.3, and Theorem 3.9.5).

PROPOSITION 7. *Consider the following conditions* ($E \in T_2$):

(2) *f is closed and the counter-image of any point is compact.*

(3) *f has a closed graph and the counter-image of any compact set is compact.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). *If F is a k-space, then also* (3) $\Rightarrow$ (2).

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). Let $K \subset F$ be compact and $\{a_\sigma\}$ a net in $f^{-1}(K)$. There is a subnet $\{a_{\sigma'}\}$ such that $f(a_{\sigma'}) \to c \in K$. Since f is inversely subcontinuous, there is a subnet $\{a_{\sigma''}\}$ of $\{a_{\sigma'}\}$ with $a_{\sigma''} \to a \in E$. The graph of f is closed, so that $a \in f^{-1}(c) \subset f^{-1}(K)$. This shows that $f^{-1}(K)$ is compact. Similarly it can be proved that f is closed.

(2) $\Rightarrow$ (1). Let $f(a_\sigma)\to c\in F$. Put $A_\sigma=\{a_{\sigma'}\colon \sigma'\geqslant\sigma\}$ and $B_\sigma=\bar{A}_\sigma\cap f^{-1}(c)$ for all indices σ. By (2), $f(\bar{A}_\sigma)\supset\overline{f(A_\sigma)}\ni c$ and $f^{-1}(c)$ is compact. It follows that $\{B_\sigma\}$ is a family of compact sets having the finite intersection property; there is a point b in its intersection — b is, by definition, a cluster point of the net $\{a_\sigma\}$ and $b\in f^{-1}(c)$. This yields both parts of (1).

Finally, let F be a k-space and assume condition (3). Let $A\subset E$ be closed and $K\subset F$ compact. Given $c\in\overline{f(A)\cap K}$, choose a net $\{a_\sigma\}\subset A\cap f^{-1}(K)$ such that $f(a_\sigma)\to c$. Since $A\cap f^{-1}(K)$ is compact, there is a subnet of $\{a_\sigma\}$ which tends to a point b in the set. The graph of f is closed, so that $c=f(b)\in f(A)\cap K$. This proves that the intersection of $f(A)$ with any compact set K is closed. By Theorem 3.3.18 from Engelking [6], $f(A)$ is closed.

3. Main result. A set is called *nearly open* if it is in the interior of its closure. A function is called *nearly continuous* (*nearly open*) if the counter-image (image) of any open set is nearly open (Pták [16] and [17]).

THEOREM. *Let E be a topological space, and let F be a Čech-complete topological space. A mapping f of E to F is continuous if (and only if) f is nearly continuous and has a Δ-closed graph.*

Proof. Let the letter G stand for open sets in F. Since F is a regular space and $f^{-1}(G)\subset\operatorname{Int}\overline{f^{-1}(G)}\subset\overline{f^{-1}(G)}$, it is sufficient to prove that $\overline{f^{-1}(G)}\subset f^{-1}(\bar{G})$. Put $V_0=G$ and $W_0=F\setminus\bar{G}$ and assume, to get a contradiction, that $\overline{f^{-1}(V_0)}\cap f^{-1}(W_0)\neq 0$.

Let $\{\mathscr{C}_n\}$ be a sequence of open covers of F such that any net $\{c_\sigma\colon \sigma\in\Sigma\}\subset F$ has a convergent subnet provided for each n the family $\big\{\{c_{\sigma'}\colon\sigma'\geqslant\sigma\}\colon\sigma\in\Sigma\big\}$ contains sets of diameter less than $\mathscr{C}_n$. (Such a sequence of open covers of F exists by a variant of Theorem 3.9.2 from Engelking [6] due to Frolík [7].) Following the proof of Theorem of Byczkowski and Pol [5] we can construct sequences $\{V_n\}$ and $\{W_n\}$ of open subsets of F such that

(α) $\bar{V}_{n+1}\subset V_n$ and $\bar{W}_{n+1}\subset W_n$, $n=0,1,2,\ldots$;

(β) V_n and W_n have diameter less than $\mathscr{C}_n$, $n\in N$;

(γ) $\overline{f^{-1}(V_n)}\cap f^{-1}(W_n)\neq 0$, $n\in N$.

By (γ), there exist nets $\{a_n^\tau\colon\tau\in T_n\}\subset f^{-1}(V_n)$ and elements $b_n\in f^{-1}(W_n)$ such that $a_n^\tau\xrightarrow[\tau]{} b_n$. Consider the product index set

$$\Sigma=N\times\prod_{i\in N}T_i$$

($\sigma\leqslant\sigma'$ iff $n\leqslant n'$ and $\nu(i)\leqslant\nu'(i)$ for all $i\in N$, where $\sigma=(n,\nu)$ and $\sigma'=(n',\nu')$) and the corresponding product net

$$\{(a_\sigma,b_\sigma)\colon\sigma\in\Sigma\}\subset E\times E$$

($a_\sigma=a_n^{\nu(n)}$ and $b_\sigma=b_n$ for $\sigma=(n,\nu)\in\Sigma$).

Notice that if $U_n \subset E$ are open sets with $b_n \in U_n$, then there is an index

$$\nu_0 \in \prod_{i \in N} T_i$$

such that $a_n^{\nu(n)} \in U_n$ for all $n \in N$ and $\nu \geqslant \nu_0$. The fact has two consequences. Firstly, $(a_\sigma, b_\sigma) \to \Delta_E$. Secondly, the net $\{(a_\sigma, b_\sigma)\}$ has no cluster points out of Δ_E provided E is a Hausdorff space (this will be used in Corollary 1). Since $f(a_n^{\nu(n)}) \in V_n$, (α) and (β) imply that the net $\{f(a_\sigma)\}$ has a subnet which converges to an element of V_0. We have even more: any subnet of $\{f(a_\sigma)\}$ has a subnet which converges to an element of V_0. The same holds for $\{f(b_\sigma)\}$ and W_0. Hence there exists a subnet $\{(a_{\sigma'}, b_{\sigma'})\}$ of $\{(a_\sigma, b_\sigma)\}$ such that $\big(f(a_{\sigma'}), f(b_{\sigma'})\big) \to (c, d) \in V_0 \times W_0$. Clearly, $(a_{\sigma'}, b_{\sigma'}) \to \Delta_E$. By the assumption that f has a Δ-closed graph, $(c, d) \in \Delta_F$. This gives a contradiction: $c = d \in V_0 \cap W_0 = G \cap (F \setminus \bar{G}) = 0$.

4. Consequences. A mapping f of E to F is called *perfect* if E is a Hausdorff space and f is continuous and satisfies condition (2) (cf. Engelking [6], Section 3.7).

COROLLARY 1 (cf. Byczkowski and Pol [5], Theorem and Corollary). *Let E be a Hausdorff topological space, and let F be a Čech-complete topological space. A mapping f of E to F is perfect if (and only if) f is nearly continuous and satisfies condition (2) (equivalently, (1) or (3)).*

Proof. If E is a Urysohn space, the assertion follows immediately from the Theorem and Propositions 5 and 7. If E is a Hausdorff space, we must apply the Lemma – instead of the assumption that f has a Δ-closed graph – in the final part of the proof of the Theorem.

COROLLARY 2 ([5]). *Let E and F be as in Corollary 1. An open bijection f of E onto F is continuous if (and only if) f is nearly continuous.*

COROLLARY 3 ([5]). *Let E and F be Čech-complete topological spaces. A mapping f of E to F is continuous if (and only if) f is nearly continuous and has a closed graph.*

Proof. The graph $G(f) \subset E \times F$ is Čech-complete in its relative product topology (cf. Engelking [6], Theorems 3.9.8 and 3.9.6). The induced mapping φ of E onto $G(f)$, defined by $\varphi(x) = \big(x, f(x)\big)$, fulfills the assumptions of Corollary 2. Hence φ and f are continuous.

The next corollary, which is a consequence of the Theorem and Proposition 6, extends the corresponding result of Kelley [10] (Problem 6.R: for F completely metrizable left-complete, and for F locally compact).

COROLLARY 4. *Let E be a topological group, and let F be a Čech-complete topological group. A homomorphism f from E to F is continuous if (and only if) f is nearly continuous and has a closed graph.*

Now we pass on to open mapping theorems. Corollary 2 yields

COROLLARY 5 ([5]). *Let E be a Hausdorff topological space, and let F be a Čech-complete topological space. A continuous bijection g of F onto E is open if (and only if) g is nearly open.*

Corollary 3 yields

COROLLARY 6 ([5]). *Let E and F be Čech-complete topological spaces. A bijection g of F onto E is open if (and only if) g is nearly open and has a closed graph.*

Corollaries 5 and 6 are not true for mappings which are not one-to-one; Byczkowski and Pol [5] gave an example of a continuous nearly open surjection $g: F \to E$ which is not open, where F is separable and completely metrizable and E is the unit interval. We are only able to replace the assumptions that g be an injection with the assumption that the quotient space $F/R(g)$ be Čech-complete and the quotient map $q: F \to F/R(g)$ be open (where the equivalence relation $R(g)$ is the set of all pairs $(a, b) \in F \times F$ for which $g(a) = g(b)$). Our final result shows that, for homomorphisms between topological groups, the situation is much better.

COROLLARY 7. *Let E be a topological group, and let F be a Čech-complete topological group. A homomorphism g from F to E, with a closed kernel, is open if and only if g is nearly open and has a closed graph.*

Proof. The kernel $\mathrm{Ker}\, g$ is an invariant subgroup of F, and the image $E_0 = g(F)$ is a subgroup of E. Consider the quotient topological group $F_0 = F/\mathrm{Ker}\, g$, the canonical quotient mapping $q: F \to F_0$, and the induced mapping $g_0: F_0 \to E$. As well known, q is a continuous *open* homomorphism and g_0 is a monomorphism.

Necessity. Assume g is open. Then E_0 is open and closed, g_0 is open, $g_0^{-1}: E_0 \to F_0$ continuous. Let $F \ni y_\sigma \to y \in F$ and $g(y_\sigma) \to x \in E$. Then $x \in E_0$, $q(y_\sigma) \to q(y)$, and $q(y_\sigma) = g_0^{-1}(g(y_\sigma)) \to g_0^{-1}(x)$. Hence $x = g_0(q(y)) = g(y)$. Thus g has a closed graph.

Sufficiency. Assume g is nearly open and has a closed graph. (The last assumption yields that $\mathrm{Ker}\, g$ is closed.) By Brown [4] (Theorem 2 and Corollary 3 to Theorem 1), F_0 is Čech-complete and complete in its two-sided uniformity. Since g is nearly open and q continuous, g_0^{-1} is nearly continuous. Since g has a closed graph and q is open, one can prove that g_0^{-1} has a closed graph. By Corollary 4, g_0^{-1} is continuous. Thus g_0 is open when considered as a map to E_0; $g = g_0 \circ q$ is open as a map to E_0. Now, it is sufficient to prove that E_0 is open in E. Since $E_0 = g(F)$ is nearly open in E, it is sufficient to verify that E_0 is closed in E. Let $E_0 \ni x_\sigma \to x \in E$. The net $\{g_0^{-1}(x_\sigma)\}$ is fundamental in F_0 with its two-sided uniformity, because g_0^{-1} is a continuous isomorphism. Let $g_0^{-1}(x_\sigma) \to y_0 \in F_0$.

Since g_0 has a closed graph (as a subset of $F_0 \times E$), $x = g_0(y_0) \in E_0$. This completes the proof.

Corollary 7 contains the corresponding results of Kelley [10] (Problem 6.R: for F completely metrizable left-complete, and for F locally compact) and Brown [4] (Theorem 4: for g continuous). It shows, in particular, that every Fréchet space (= completely metrizable locally convex topological vector space) is a Pták space (= B-complete space in [16] = fully complete space), and that every Čech-complete topological group is a $B(\mathscr{A})$-group in the sense of Husain [9], Chapter V.

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