COLLOQUIUM MATHEMATICUM

VOL. XLI 1979 FASC. 2

CARDINAL FUNCTIONS ON HYPERSPACES

BY

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For a space X, $\mathscr{C}(X)$ denotes the collection of all non-empty compact subsets of X, and $\mathscr{A}(X)$ denotes the collection of all non-empty closed subsets of X. It is well known that $\mathscr{C}(X)$ and $\mathscr{A}(X)$ can be topologized by the finite topology a base of which consists of all sets of the form

$$\langle U_1, \ldots, U_n\rangle = \{K \in \mathscr{C}(X) \text{ (respectively, } K \in \mathscr{A}(X)): \\ K \subset \bigcup U_j \text{ and } K \cap U_j \neq \emptyset \text{ for each } j\},$$

where $U_1, \ldots, U_n$ are open sets of X. We shall consider in this paper the heredity on $\mathscr{C}(X)$ or $\mathscr{A}(X)$ of cardinal functions on X.

Let $\chi, \psi, \partial, d, \omega, \pi, z$ be the cardinal functions called *character*, *pseudo-character*, *tightness*, *density*, *weight*, *π-weight*, *width*, respectively, as defined in [2]. Let φ be the cardinal function of χ and ψ; we put

$$\varphi_K(X) = \sup\{\varphi(K, X): K \in \mathscr{C}(X)\}, \qquad \varphi_A(X) = \sup\{\varphi(F, X): F \in \mathscr{A}(X)\}$$

(for $\varphi(K, X)$ and $\varphi(F, X)$, see [2]).

Let φ be the cardinal function of d and π; we set

$$\varphi_K(X) = \sup\{\varphi(K): K \in \mathscr{C}(X)\} \quad \text{and} \quad \varphi_A(X) = \sup\{\varphi(F): F \in \mathscr{A}(X)\}.$$

In the sequel, all spaces are assumed to be Hausdorff and all cardinals are assumed to be infinite. N always denotes the set of all positive integers.

LEMMA 1. $\psi(\mathscr{C}(X)) \geqslant \pi_K(X)$.

Proof. Let $\psi(\mathscr{C}(X)) = m$ and $K \in \mathscr{C}(X)$. Then K can be represented as

$$K = \bigcap\{\langle W_{a1}, \ldots, W_{ak_a}\rangle: a \in A\},$$

where $k_a \in N$ and $|A| \leqslant m$. Put

$$\mathscr{V} = \{V_{aj} = K \cap W_{aj}: j = 1, \ldots, k_a, a \in A\}.$$

We shall show that $\mathscr{V}$ is a π-base for the subspace K. Let $\emptyset \neq G' = G \cap K$ with G open in X. Then

$$L = K \setminus G \in \mathscr{C}(X) \quad \text{and} \quad K \neq L.$$

Thus there exists an $a \in A$ such that $L \notin \langle W_{a1}, \dots, W_{ak}\rangle$. This implies $L \cap W_{aj} = \emptyset$ for some j. It follows that $G' \subset V_{aj}$. Hence $\pi(K) \leqslant m$ for each $K \in \mathscr{C}(X)$.

LEMMA 2. *If X is locally compact, then* $\partial(\mathscr{C}(X)) \geqslant \chi_K(X)$.

Proof. Let X be a locally compact space with $\partial(\mathscr{C}(X)) = m$. Let $K \in \mathscr{C}(X)$. Put

$$\mathscr{U} = \{U\colon\ U \text{ is a compact neighborhood of } K \text{ in } X\}.$$

It follows from the local compactness of X that $K \in \mathrm{Cl}_{\mathscr{C}(X)}(\mathscr{U})$. Since $\partial(\mathscr{C}(X)) = m$, there exists a subcollection $\mathscr{U}_0$ of $\mathscr{U}$ with $|\mathscr{U}_0| \leqslant m$ such that $K \in \mathrm{Cl}_{\mathscr{C}(X)}(\mathscr{U}_0)$. To see that $\mathscr{U}_0$ is a neighborhood base of K in X, suppose that $K \subset G$ with G open in X. Then $K \in \langle G\rangle$. There exists a $U \in \mathscr{U}_0$ such that $U \in \langle G\rangle$, which implies $U \subset G$. Therefore, $\{\mathrm{Int}\, U\colon\ U \in \mathscr{U}_0\}$ is a local base of K in $\mathscr{C}(X)$.

In a similar way we have

COROLLARY. *If X is regular, then* $\partial(\mathscr{A}(X)) \geqslant \chi_K(X)$.

LEMMA 3. $\partial(\mathscr{C}(X)) \geqslant d_K(X)$.

Proof. Let $\partial(\mathscr{C}(X)) = m$. Then

$$\mathscr{A}(K) \subset \mathscr{C}(X) \quad \text{and} \quad \partial(\mathscr{A}(K)) \leqslant m.$$

Thus it suffices to prove the inequality $\partial(\mathscr{A}(X)) \geqslant d(X)$ when X is compact. But this is clear from Theorem 1 in [3].

THEOREM 1. $\chi(\mathscr{C}(X)) = \chi_K(X)\, d_K(X)$.

Proof. Let $\chi(\mathscr{C}(X)) = m$. By Lemma 1 and by trivial inequalities $\psi \leqslant \chi$ and $d \leqslant \pi$ we have $d_K(X) \leqslant m$. Let $K \in \mathscr{C}(X)$ and let $\mathscr{B}(K) = \{\mathscr{B}_a\colon\ a \in A\}$ be a local base of K in $\mathscr{C}(X)$ with $|A| \leqslant m$. Without loss of generality we can assume that

$$\mathscr{B}_a = \langle B_{a1}, \dots, B_{ak_a}\rangle,$$

where each B_{aj} is open in X and $k_a \in N$. Then we put

$$B_a = \bigcup \{B_{aj}\colon\ j = 1, \dots, k_a\}.$$

We infer easily that $\{B_a\colon\ a \in A\}$ is a base of K in X, and thus $\chi_K(X) d_K(X) \leqslant m$. Conversely, let $m = \chi_K(X) d_K(X)$. Then we show that $\chi(\mathscr{C}(X)) \leqslant m$. Since $\chi_K(X) \leqslant m$, there exists a base $\{U_a\colon\ 0 \leqslant a < \omega_m\}$ of K in X, where ω_m is an initial ordinal of m. On the other hand, $d_K(X) \leqslant m$ implies the existence of a dense set $D = \{d_a\colon\ 0 \leqslant a < \omega_m\}$. Since $\chi(p, X) \leqslant m$, there exists a local base $\{V_a(d_\beta)\colon\ 0 \leqslant a < \omega_m\}$ of d_β in X. Put

$$\mathscr{V} = \{V_a(d_\beta)\colon\ 0 \leqslant a, \beta < \omega_m\}.$$

Then $|\mathscr{V}| \leqslant m$. Let $\Delta(\mathscr{V})$ be the totality of finite subcollections of $\mathscr{V}$. For each $\{V_1, \ldots, V_k\} \in \Delta(\mathscr{V})$ and $0 \leqslant a < \omega_m$ we put

$$\langle V_1, \ldots, V_k : U_a\rangle = \{L \in \mathscr{C}(X): L \subset U_a \text{ and } L \cap V_j \neq \emptyset \text{ for each } j\}.$$

Set

$$\langle \mathscr{V}\rangle = \{\langle V_1, \ldots, V_k: U_a\rangle : \{V_1, \ldots, V_k\} \in \Delta(\mathscr{V}), 0 \leqslant a < \omega_m\}.$$

Then $\langle \mathscr{V}\rangle$ is a family of open sets containing K with $|\langle \mathscr{V}\rangle| \leqslant m$. Let $\langle W_1, \ldots, W_n\rangle$ be an open set containing K. Since $K \subset \bigcup W_j$, we have $U_a \subset \bigcup W_j$ for some a. Now, $W_j \cap K \neq \emptyset$ implies that there exists a $V_j \in \mathscr{V}$ such that $V_j \subset W_j$. Thus

$$K \in \langle V_1, \ldots, V_n : U_a\rangle \subset \langle W_1, \ldots, W_n\rangle.$$

Hence we have $\chi(\mathscr{C}(X)) \leqslant m$.

COROLLARY 1. *$\mathscr{C}(X)$ is first countable if and only if every compact set of X is separable and of countable character.*

This is an answer to Question 2 proposed by Smithson in [4]. The similar argument is applicable to the case $\mathscr{A}(X)$, and so we have

COROLLARY 2. $\chi(\mathscr{A}(X)) = \chi_A(X) d_A(X)$.

COROLLARY 3. *If X is locally compact, then $\chi(\mathscr{C}(X)) = \partial(\mathscr{C}(X))$.*

Proof. The inequality $\partial(\mathscr{C}(X)) \leqslant \chi(\mathscr{C}(X))$ is trivial. Conversely, by Lemmas 2 and 3, we have $\chi_K(X)$, $d_K(X) \leqslant \partial(\mathscr{C}(X))$. Hence by Theorem 1 we have the equality.

COROLLARY 4. *$\chi(\mathscr{C}(X)) \leqslant \partial(\mathscr{A}(X))$ if X is regular.*

For the proof use the Corollary to Lemma 2.

This refines Corollary 2 to Theorem 2 in [3].

THEOREM 2. $\psi(\mathscr{C}(X)) = \psi_K(X)\pi_K(X)$.

Proof. By Lemma 1, $m = \psi(\mathscr{C}(X)) \geqslant \pi_K(X)$. Let $K \in \mathscr{C}(X)$. Write

$$K = \bigcap\{\langle W_{a1}, \ldots, W_{ak}\rangle : a \in A\}, \quad |A| \leqslant m.$$

Set $W_a = \bigcup\{W_{aj}: j = 1, \ldots, k_a\}$. If $p \notin K$, then

$$L = \{p\} \cup K \in \mathscr{C}(X) \quad \text{and} \quad L \neq K.$$

Therefore, $L \notin \langle W_{a1}, \ldots, W_{ak}\rangle$ for some a. Then $p \notin W_a$. Consequently $K = \bigcap W_a$. Hence we have

$$\psi(\mathscr{C}(X)) \geqslant \psi_K(X)\pi_K(X).$$

Conversely, let $m = \psi_K(X)\pi_K(X)$ and $K \in \mathscr{C}(X)$. Let $\mathscr{G} = \{G_a: a \in A\}$ be a π-base for K and let V_a be an open set of X with $V_a \cap K = G_a$. Put

$$\mathscr{V} = \{V_a: a \in A\} \quad \text{and} \quad K = \bigcap\{W_\beta: \beta \in B\}, \ |B| \leqslant m.$$

Let $\Delta(\mathscr{V})$ be the totality of finite subcollections of $\mathscr{V}$. Then $|\Delta(\mathscr{V})| \leqslant m$. For each $\{V_1, \ldots, V_k\} \in \Delta(\mathscr{V})$ and $\beta \in B$, construct

$$\langle V_1, \ldots, V_k \colon W_\beta \rangle = \{L \in \mathscr{C}(X) \colon L \subset W_\beta, L \cap V_j \neq \emptyset \text{ for each } j\}.$$

Then $\langle V_1, \ldots, V_k \colon W_\beta \rangle$ is an open set containing K. Suppose that $K \neq L \in \mathscr{C}(X)$. If $L \setminus K \neq \emptyset$, then there exist a point $p \in L \setminus K$ and $\beta \in B$ with $p \notin W_\beta$. Then

$$L \notin \langle V_1, \ldots, V_k \colon W_\beta \rangle$$

for any finite collection $\{V_1, \ldots, V_k\}$ of $\mathscr{V}$. If $L \subset K$, then there exist a point $p \in K \setminus L$ and G_a with $G_a \cap L = \emptyset$. In this case we have $L \notin \langle V_a \colon W_\beta \rangle$ for any $\beta \in B$. Therefore, it follows that

$$K = \bigcap \{\langle V_1, \ldots, V_k \colon W_\beta \rangle \colon \{V_1, \ldots, V_k\} \in \Delta(\mathscr{V}), \beta \in B\}.$$

This completes the proof.

A similar argument shows that

COROLLARY 1. $\psi(\mathscr{A}(X)) = \psi_A(X)\pi_A(X)$.

COROLLARY 2. *Every point in $\mathscr{C}(X)$ is G_δ if and only if every compact set of X is G_δ and has a countable π-base.*

THEOREM 3. $d(X) = d(\mathscr{C}(X))$.

Proof. Suppose that $d(\mathscr{C}(X)) = m$. Then there exists a dense subset $\{K_a \colon a \in A\}$ of $\mathscr{C}(X)$ with $|A| = m$. Take an arbitrary point $p_a \in K_a$ for each a. Then $D = \{p_a \colon a \in A\}$ is dense in X. Hence $d(X) \leqslant m$.

Conversely, suppose that D is a dense set of X with $|D| = m$. Then $\mathscr{F}(D) = \{E \subset D \colon E \text{ finite}\}$ is a dense subset of $\mathscr{C}(X)$. Hence $d(\mathscr{C}(X)) \leqslant m$.

COROLLARY 1 (Theorem 2.1 in [1]). $|\mathscr{C}(X)| \leqslant 2^{2^{d(X)}}$.

For the proof, use 2.4 in [2].

COROLLARY 2. $|\mathscr{C}(X)| \leqslant 2^{z(X)\chi_K(X)}$.

Indeed, by 2.6, 2.20 in [2] and Theorem 1 we have

$$|\mathscr{C}(X)| \leqslant d(\mathscr{C}(X))^{\chi(\mathscr{C}(X))} = d(X)^{\chi_K(X)d_K(X)} \leqslant z(X)^{z(X)\chi_K(X)} = 2^{z(X)\chi_K(X)}.$$

THEOREM 4. (i) $\pi(X) = \pi(\mathscr{C}(X))$. (ii) $\omega(X) = \omega(\mathscr{C}(X))$.

The proof is trivial.

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Reçu par la Rédaction le 1. 8. 1977