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A UNIVERSAL NON-COMPACT OPERATOR

BY

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In this note X, Y, Z, and W are Banach spaces and "operator" means "bounded linear operator". An operator $T\colon X \to Y$ is said to factor through an operator $U\colon Z \to W$ iff there are operators $A\colon X \to Z$ and $B\colon W \to Y$ such that $BUA = T$.

Lindenstrauss and Pełczyński [3] showed that the sum operator $\sigma\colon l_1 \to m$ factors through every non-weakly compact operator. (m is the space of bounded scalar-valued sequences with the sup norm. σ is defined by $\sigma(\{a_i\}_{i=1}^{\infty}) = \{\sum_{i=1}^{n} a_i\}_{n=1}^{\infty}$.) Thus σ is a universal non-weakly compact operator in the sense that an operator is not weakly compact iff σ factors through it. Similarly, the class of non-compact operators has a universal element.

THEOREM. *Let T be an operator from X to Y. T is not compact iff the formal identity operator $I\colon l_1 \to m$ factors through T.*

Proof. Suppose that T is not compact. Suppose first that T is weakly compact. It then follows that there is $\varepsilon > 0$ and a sequence $\{x_i\}_{i=1}^{\infty}$ in X such that $\|x_i\| = 1$, $\|T(x_i)\| \geqslant \varepsilon$, and $\{T(x_i)\}_{i=1}^{\infty}$ converges weakly to 0. By [1], Theorem 3, there is a subsequence $\{z_i\}_{i=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ such that $\{T(z_i)\}_{i=1}^{\infty}$ is a basic sequence. Define $A\colon l_1 \to X$ by

$$A(\{a_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} a_i z_i$$

and define $B\colon \operatorname{clsp}\{T(z_i)\}_{i=1}^{\infty} \to m$ by

$$B\Big(\sum_{i=1}^{\infty} b_i T(z_i)\Big) = \{b_i\}_{i=1}^{\infty},$$

where $\operatorname{clsp}\{T(z_i)\}_{i=1}^{\infty}$ denotes the closure of the linear subspace generated by the sequence $\{T(z_i)\}_{i=1}^{\infty}$. A is well defined because $\{z_i\}_{i=1}^{\infty}$ is bounded; B is well defined because $\{T(z_i)\}_{i=1}^{\infty}$ is bounded away from 0. The closed graph theorem shows that A and B are continuous.

Since m is a $\mathscr{P}_1$ space (see [2], p. 94, for the definition and basic properties of $\mathscr{P}_1$ spaces), there is an operator $B'\colon Y \to m$ such that the restriction of B' to $\operatorname{clsp}\{T(z_i)\}_{i=1}^{\infty}$ is B. Clearly $B'TA = I$, so I factors through T.

Now suppose that T is not weakly compact. Since $\sigma\colon l_1 \to m$ factors through T, it is sufficient to show that $I\colon l_1 \to m$ factors through σ. Let $A\colon l_1 \to l_1$ be the identity map and define $B\colon m \to m$ by

$$B(\{a_i\}_{i=1}^{\infty}) = \{a_1, a_2 - a_1, a_3 - a_2, \ldots\}.$$

Then A and B are continuous and $I = B\sigma A$.

The converse is trivial.

Remark. If T is not compact and Y is separable, then the formal identity operator $I'\colon l_1 \to c_0$ factors through T. Indeed, by the theorem there are operators $A\colon l_1 \to X$ and $B\colon Y \to m$ such that $BTA = I$. Since Y is separable, the closure of the range of B is separable, so by a result of Sobczyk (see e.g., [4], Theorem 4) there is a projection P from the closure of the range of B onto c_0. $I' = (PB)TA$, so I' factors through T.

REFERENCES

[1] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Mathematica 17 (1958), p. 151-164.

[2] M. M. Day, *Normed linear spaces*, 1958.

[3] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathscr{L}_p$-spaces and their applications*, Studia Mathematica 29 (1968), p. 275-326.

[4] A. Pełczyński, *Projections in certain Banach spaces*, ibidem 19 (1960), p. 209-228.

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