COLLOQUIUM MATHEMATICUM

VOL. XXXVI 1976 FASC. 2

ON 2-CELL IMBEDDINGS OF COMPLETE n-PARTITE GRAPHS

BY

H. V. KRONK (BINGHAMTON, NEW YORK),
R. D. RINGEISEN* (HAMILTON, NEW YORK)
AND A. T. WHITE** (KALAMAZOO, MICHIGAN)

In this paper we first survey the known results concerning 2-cell imbeddings of complete n-partite graphs into closed orientable 2-manifolds. This discussion of necessity involves both the genus and the maximum genus parameters. Many of the known values of these parameters are for regular complete n-partite graphs, which are all Cayley graphs; we discuss Jacques' theory of reduced constellations, which applies to 2-cell imbeddings of Cayley graphs. Application of the theory is illustrated, in constructing both genus and maximum genus imbeddings, for the complete 3-partite graphs $K_{m,m,m}$ (m odd). Finally, we show that all complete n-partite graphs are upper-imbeddable.

1. Introduction. A graph G is said to be *imbedded* in the closed orientable 2-manifold M if the geometric realization of G, as a finite 1-complex, is homeomorphic to a subspace of M. If M has genus k (k a non-negative integer), we write $M = S_k$. The *genus* $\gamma(G)$ of a graph G is the minimum k such that G imbeds in S_k. An imbedding of G into $S_{\gamma(G)}$ is called a *minimal imbedding*. The components of $S_k - G$ are called *regions*, and a region is said to be a *2-cell* if it is homeomorphic to R^2. A *2-cell imbedding* of G in S_k has every region a 2-cell. It is well known (see, for example, König [9] or Youngs [20]) that a minimal imbedding of a connected graph G must be 2-cell. Any 2-cell imbedding of a graph G, with p vertices and q edges, into S_k with r regions, must satisfy

$$p - q + r = 2 - 2k.$$

We now define the *maximum genus* $\gamma_M(G)$ of a connected graph G to be the maximum k such that G 2-cell imbeds in S_k. From a result of

* Research supported in part by the Colgate University Research Council, where the author was formerly a faculty member; the author is currently at Purdue University, Fort Wayne.

** Research supported in part by National Science Foundation Grant No. GP-36547.

Duke [2] it now follows that G has a 2-cell imbedding in S_k if and only if $\gamma(G) \leqslant k \leqslant \gamma_M(G)$. From the extended Euler formula given above it readily follows that

$$\gamma_M(G) \leqslant \left[\frac{\beta(G)}{2}\right], \quad \text{where } \beta(G) = q-p+1.$$

If equality holds, G is said to be *upper-imbeddable.* This requires a 2-cell imbedding with $r = 1$ (if $\beta(G)$ is even) or with $r = 2$ (if $\beta(G)$ is odd).

Attention was focused on the genus parameter by Ringel and Youngs (see, for example, [14]), as their solution to the Heawood Map-Coloring Conjecture involved the determination of $\gamma(K_n)$. Other genus investigations have tended to center on graphs related to K_n, as indicated in the following definition. For $n \geqslant 2$, a *complete n-partite graph* $G = K_{m_1,m_2,\ldots,m_n}$ has $p = \sum_{i=1}^{n} m_i$ vertices partitioned into n partite sets of m_i $(1 \leqslant i \leqslant n)$ mutually non-adjacent vertices, respectively, and all edges joining vertices from distinct partite sets. Thus

$$q = \sum_{i<j} m_i m_j$$

for this graph. If $m_i = m$ for $1 \leqslant i \leqslant n$, the graph is regular, and we write $G = K_{n(m)}$. Equivalently, $G = K_{n(m)}$ can be defined by specifying $\bar{G} = nK_m$; that is, the complement consists of n disjoint copies of the complete graph K_m. Note that, in this notation, $K_n = K_{n(1)}$.

2. Known results. The following genus formulae have been established for complete n-partite graphs.

THEOREM 2.1 (Ringel and Youngs [14]).

$$\gamma(K_n) = \left\{\frac{(n-3)(n-4)}{12}\right\} \quad \textit{for } n \geqslant 3.$$

THEOREM 2.2 (Ringel [12]).

$$\gamma(K_{m_1,m_2}) = \left\{\frac{(m_1-2)(m_2-2)}{4}\right\} \quad \textit{for } m_1, m_2 \geqslant 2.$$

THEOREM 2.3 (Ringel and Youngs [15]).

$$\gamma(K_{3(m)}) = \frac{(m-2)(m-1)}{2}.$$

THEOREM 2.4 (White [17]).

$$\gamma(K_{mn,n,n}) = \frac{(mn-2)(n-1)}{2}.$$

THEOREM 2.5 (White [16]).

$$\gamma(K_{m_1,m_2,m_3}) = \left\{\frac{(m_1-2)(m_2+m_3-2)}{4}\right\}$$

for $m_1 \geqslant m_2 \geqslant m_3$, $m_1 \geqslant 2$, *and* $m_2+m_3 \leqslant 6$.

The recently settled conjecture $\gamma(K_{4(m)}) = (m-1)^2$ is readily verified for $m = 1$ and 2; Ringel [13] first verified the case $m = 4$.

THEOREM 2.6 (Garman [4]). $\gamma(K_{4(m)}) = (m-1)^2$ *for* $m \equiv 2 \pmod 4$.

THEOREM 2.7 (Jungerman [8]). $\gamma(K_{4(m)}) = (m-1)^2$ *if and only if* $m \neq 3$.

Since an imbedding of $K_{4(3)}$ on S_5 has been found, the issue is now completely resolved.

The graph $K_{n(2)}$ has been called the *n-dimensional octahedral graph.* The conjecture here is that

$$\gamma(K_{n(2)}) = \left\{\frac{(n-3)(n-1)}{3}\right\};$$

this is readily verified for $n \leqslant 4$. White [19] has shown that $\gamma(K_{6(2)}) = 5$, and Garman (private communication) has found that the conjecture is asymptotically correct for n a power of 3. We have also

THEOREM 2.8 (Gross and Alpert [5]).

$$\gamma(K_{n(2)}) = \frac{(n-3)(n-1)}{3} \quad \text{for } n \equiv 4 \pmod 6.$$

For the maximum genus of complete n-partite graphs, the following results appear in the literature.

THEOREM 2.9 (Nordhaus et al. [10]).

$$\gamma_M(K_n) = \left[\frac{(n-2)(n-1)}{4}\right].$$

THEOREM 2.10 (Ringeisen [11]).

$$\gamma_M(K_{m_1,m_2}) = \left[\frac{(m_1-1)(m_2-1)}{2}\right].$$

For 2-cell imbeddings of connected graphs G on S_k, where $\gamma(G) < k < \gamma_M(G)$, those which are *self-dual* (i.e. the geometric dual is isomorphic to G) are of special interest. The following theorem appears in [19].

THEOREM 2.11. *If* $m(n-1) \equiv 0 \pmod 4$, *then* $K_{n(m)}$ *has a self-dual imbedding.*

Thus K_n ($n \equiv 1 \pmod 4$), $K_{n(2)}$ (n odd), and $K_{m,m}$ ($m \equiv 0 \pmod 4$) all have self-dual imbeddings. Theorem 2.11, together with the extended

Euler formula, shows that, for $n \equiv 3 \pmod 4$, $K_{n(m)}$ has a self-dual imbedding if and only if m is even.

The proof of Theorem 2.11, as well as that of several of the other theorems given above, was greatly facilitated by the circumstance that $K_{n(m)}$ is, in fact, a Cayley graph; the imbedding theory outlined below (due to Jacques; see [7] or [18] for a more thorough treatment) applies to such situations.

3. Theory of reduced constellations. Let Γ be a finite group, generated by a set $\Delta \subseteq \Gamma$. The *Cayley color graph* $C_\Delta(\Gamma)$ has vertex set Γ, and (g, g') is a directed edge — labeled (or colored) with $\delta \in \Delta$ — if and only if $g' = g\delta$. Now let $\Delta^{-1} = \{\delta^{-1} \mid \delta \in \Delta\}$, and form $\Delta^* = \Delta \cup \Delta^{-1}$; the elements of Δ^* are called *currents*. We assume that only generators of order two are contained in $\Delta \cap \Delta^{-1}$, and for such a δ we adopt the standard convention that the two directed edges $(g, g\delta)$ and $(g\delta, g)$ can be represented as a single undirected edge $[g, g\delta]$, labeled with δ. The *Cayley graph* $G_\Delta(\Gamma)$ is obtained from $C_\Delta(\Gamma)$ by deleting all labels (colors) and arrows (directions) from the edges; $G_\Delta(\Gamma)$ is a graph having all edges of the form $[g, g\delta]$ for $g \in \Gamma$ and $\delta \in \Delta$. Each regular complete n-partite graph $K_{n(m)}$ is a Cayley graph, as we will see in the next section.

Now let $G_\Delta(T)$ be 2-cell imbedded in S_k; we study the corresponding 2-cell imbedding of $C_\Delta(\Gamma)$ in S_k. This imbedding is described algebraically (see Edmonds [3] and Youngs [20]) by specifying, for each $g \in \Gamma$, the cyclic permutation σ_g of $g\Delta^*$ (the set of vertices adjacent with g) determined by the orientation on S_k. Let σ_g^* be the cyclic permutation of Δ^* induced by the action of σ_g on $g\Delta^*$, and suppose that Ω is a subgroup of Γ such that if $\Omega h = \Omega h'$, then $\sigma_h^* = \sigma_{h'}^*$ (for all $h, h' \in \Gamma$). Such a subgroup always exists, as we can set $\Omega = \{e\}$, where e is the identity of Γ. We will desire, however, to take Ω as large as possible. In the terminology of Jacques [7], Ω determines a *quotient constellation* C' for the *constellation* C ($C = C_\Delta(\Gamma)$ in S_k); C' is an imbedding of the Schreier coset graph (see [1]) for Ω in Γ, the imbedding being determined by the collection $\{\sigma_h^*\}$, taken over any set $\{h\}$ of right coset representatives of Ω in Γ. The dual $(C')^*$ of the quotient constellation is called the *reduced constellation*.

The reduced constellation is a 2-cell imbedding of a pseudograph K (with each edge directed and labeled with the current of its dual edge) in a surface S_j ($j \leqslant k$), called by Youngs [21] the *quotient graph* and *quoient manifold* for $C_\Delta(\Gamma)$ and Ω, respectively. Youngs arrives at $(C')^*$ by a different route; he first takes the dual C^* of $C_\Delta(\Gamma)$ in S_k and then "mods out" regions with identically labeled boundaries, in accordance with the subgroup Ω. Jacques' approach is consistent with that of Gustin [6]. The following theory was introduced by Gustin, developed by Youngs, and unified by Jacques [7].

Define (after Jacques) a *brin* to be an ordered pair $(g, g\delta^*)$, where g is a vertex in C, C', or $(C')^*$, and $\delta^* \in \Delta^*$. We think of each edge (in C, C', or $(C')^*$) as giving rise to two opposing brins, one in the oriented boundary of each region bordered by the edge. Then $(C')^*$ satisfies the following properties:

(i) Each brin carries a current from Δ^*.

(ii) Two opposing brins $x = (g, g\delta^*)$ and $x^{-1} = (g\delta^*, g)$ carry inverse currents; if $x = x^{-1}$, the current has order two.

(iii) The regions are in one-to-one correspondence with the right cosets of Ω in Γ.

(iv) For each region, the currents appearing in the region boundary are in one-to-one correspondence with Δ^*.

(v) If a brin x appears in the boundary of a region associated with Ωg and if x^{-1} appears in the boundary of a region associated with $\Omega g'$, then the current carried by x is in the set $g^{-1}\Omega g'$.

What is especially important for the study of imbedding problems is the converse: a reduced constellation $M(\Gamma/\Omega) = (K \text{ in } S_j)$ for $C_\Delta(\Gamma)$ and Ω satisfying properties (i)-(v) determines a 2-cell imbedding C of $C_\Delta(\Gamma)$ in S_k such that $(C')^* = M(\Gamma/\Omega)$. In fact, the region boundaries for $(C')^*$ determine the Emonds' permutation scheme for the Schreier coset graph imbedded as C', and hence for $C_\Delta(\Gamma)$ in S_k.

Now, for each vertex v of K, let π_v denote the product of the currents directed away from v, in the order of the orientation, and let ν_v be the order of π_v in Γ (π_v is determined up to conjugacy, so that ν_v is well defined); ν_v is called the *valence* of v. The *length* of a region is the length of the closed walk bounding the region. Then we have

THEOREM 3.1. *Each vertex v of degree d and valence ν in $M(\Gamma/\Omega)$ determines $|\Omega|/\nu$ regions of length $d\nu$ in the imbedding of $C_\Delta(\Gamma)$ in S_k.*

Moreover, as shown in [19], we have

THEOREM 3.2. *Let C in S_k be represented by $(C')^* = M(\Gamma/\Omega)$ in S_j with $\nu_1, \nu_2, \ldots, \nu_{p'}$ the valences of the vertices of $(C')^*$. Then*

$$k = |\Omega|(j-1)+1+\frac{|\Omega|}{2}\sum_{i=1}^{p'}\left(1-\frac{1}{\nu_i}\right).$$

We now indicate how this theory applies to 2-cell imbeddings of regular complete n-partite graphs.

4. Applications to $K_{n(m)}$. To apply the theory of the preceding section, we must first show that $K_{n(m)}$ is a Cayley graph; that is, we must find a group Γ and a generating set Δ for Γ giving $G_\Delta(\Gamma) = K_{n(m)}$. We let Z_s denote the cyclic group of order s.

THEOREM 4.1. *Let $\Gamma = Z_{nm}$ and choose $\Delta^* = \Gamma$ less all multiples of n; then $G_\Delta(\Gamma) = K_{n(m)}$.*

Proof. Consider $\bar{G}$, the complement of $G = G_\Delta(\Gamma)$. Since $n \in \Gamma$ has order m and $n \notin \Delta$, n determines n disjoint cycles of length m in $\bar{G}$. The remaining multiplies of n determine all possible chords for these cycles, so that $\bar{G} = nK_m$; that is, $G = K_{n(m)}$.

We remark that non-cyclic groups might also be chosen; however, for our present purposes $\Gamma = Z_{nm}$ will suffice.

To illustrate the application of the theory, we compute $\gamma(G)$ and $\gamma_M(G)$ for $G = K_{3(m)}$, m odd. (The computation of $\gamma(G)$ differs in method from those given in [15] and [17]; the result for $\gamma_M(G)$ is new.)

THEOREM 4.2. $\gamma(K_{3(m)}) = (m-2)(m-1)/2$ *for m odd.*

Proof. For $m = 1$, $K_{3(1)} = K_3$ and the result is trivial. For $m \geqslant 3$, let $H_m = G_\Delta(\Gamma')$ for $\Gamma' = Z_{2m}$ and $\Delta = \{1, m\}$; thus H_m is a cycle of length $2m$ with diametrically opposite vertices also adjacent. It will serve as the pseudograph K of $M(Z_{3m}/Z_m)$, after an appropriate assignment of currents from $\Delta^* = \Gamma$ less all multiples of 3, for $\Gamma = Z_{3m}$.

For example, Fig. 1 shows the case $m = 3$ with $H_3 = K_{3,3}$ in S_1. We choose $\Omega = Z_m = Z_3$ of index 3 in $\Gamma = Z_9$. With the regions and edges of this imbedding of $K_{3,3}$ labeled and directed as indicated, it is apparent that properties (i)-(v) of a reduced constellation are satisfied. Moreover, K

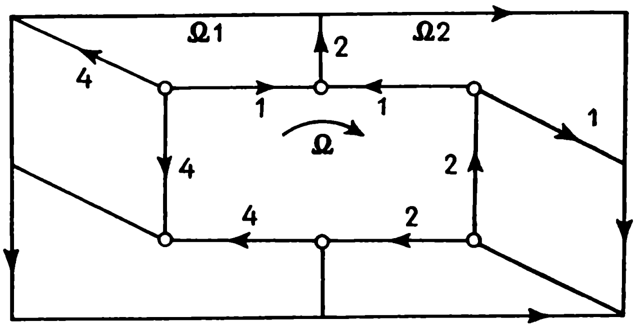

Fig. 1. $M(Z_9/Z_3)$ for $K_{3(3)}$ in S_1

also satisfies the KCL (Kirchoff Current Law; that is, ν is 1 at each vertex); thus $M(Z_9/Z_3)$ determines (6) (3) = 18 triangular regions (by Theorem 3.1) in an imbedding for $K_{3(3)}$ in S_1 (by Theorem 3.2). The imbedding itself is algebraically described by specifying the local vertex permutations $\{\sigma_i\}_0^8$: from the region boundaries of $M(Z_9/Z_3)$ we obtain

$$\sigma_0^*: (1, 8, 7, 2, 4, 5) \quad \text{for } \Omega,$$

$$\sigma_1^*: (1, 5, 7, 8, 4, 2) \quad \text{for } \Omega 1,$$

$$\sigma_2^*: (1, 2, 7, 5, 4, 8) \quad \text{for } \Omega 2,$$

so that we have

$$\sigma_0\colon (1,8,7,2,4,5), \quad \sigma_1\colon (2,6,8,0,5,3),$$
$$\sigma_3\colon (4,2,1,5,7,8), \quad \sigma_4\colon (5,0,2,3,8,6),$$
$$\sigma_6\colon (7,5,4,8,1,2), \quad \sigma_7\colon (8,3,5,6,2,0),$$
$$\sigma_2\colon (3,4,0,7,6,1),$$
$$\sigma_5\colon (6,7,3,1,0,4),$$
$$\sigma_8\colon (0,1,6,4,3,7).$$

We can now compute the counter-clockwise region boundaries for C by the rule (see [20]) that the directed edge (g_1, g_2) is followed by $(g_2, \sigma_{g_2}(g_1))$; for example, $(0,1)(1,5)(5,0)$ is one of the eighteen triangular region boundaries.

For $m \geqslant 5$, the construction generalizes that given above for $m = 3$. We begin with H_m imbedded in S_j for $j = (m-1)/2$. The three regions, each, have $2m$ boundary edges. Pick any one region (label it $\Omega = Z_m$) and label its boundary edges from Δ^* for $\Gamma = Z_{3m}$ in order as follows:

$$(1, -1, 7, -7, 13, -13, 19, -19, \ldots, 3m-5, -(3m-5)).$$

The remaining m edges can now be labeled (uniquely) so that the KCL holds. Label the region sharing the edges labeled $1, 7, 13, 19, \ldots$ $\ldots, 3m-5$ with Ω by $\Omega 1$, and label the remaining region $\Omega 2$. Properties (i)-(iii) for a reduced constellation are now trivially satisfied; verification of properties (iv) and (v) is a mild algebraic exercise. It now follows that $M(Z_{3m}/Z_m)$, which we have constructed, determines (by Theorems 3.1 and 3.2) a triangular (since $K = H_m$ is cubic and satisfies the KCL) imbedding of $K_{3(m)}$ into S_k for

$$k = m\left(\frac{m-3}{2}\right)+1 = \frac{(m-2)(m-1)}{2};$$

but triangular imbeddings are minimal.

THEOREM 4.3. *$\gamma_M(K_{3(m)}) = (3m(m-1))/2$ for m odd.*

Proof. In general, we have

$$\gamma_M(K_{3(m)}) \leqslant \left[\frac{\beta(K_{3(m)})}{2}\right] = \frac{3m(m-1)}{2},$$

with equality holding if and only if there exists a 2-cell imbedding of $K_{3(m)}$ with $r = 2$. We now construct such an imbedding. Consider the pseudograph K of Fig. 2, with currents from $\Delta = \{1, 2, 4, 5, \ldots, (3m-1)/2\}$ for $\Gamma = Z_{3m}$; then again $\Delta^* = \Gamma$ less multiples of 3, so that $G_\Delta(\Gamma) = K_{3(m)}$.

If we subdivide each edge of K, we obtain the graph $K_{2,m}$, which has an $r = 1$ 2-cell imbedding in S_j for $j = (m-1)/2$, by Theorem 2.10. Thus K has an $r = 1$ 2-cell imbedding in S_j also. This is a reduced constellation

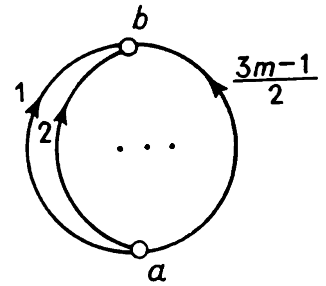

Fig. 2. K for $K_{3(m)}$

for $G_\Delta(K_{3(m)})$; the verification is immediate. It remains to compute ν for either vertex of K (since $\pi_b = \pi_a^{-1}$, $\nu_b = \nu_a$). But

$$\pi_a = \sum_{i=1}^{(3m-1)/2} i - \sum_{i=1}^{(m-1)/2} 3i = \frac{3m^2+1}{4}.$$

Now write $m = 2s+1$, so that $\pi_a = 3s^2+3s+1$, and $3m = 6s+3$; since

$$4(3s^2+3s+1)-(2s+1)(6s+3) = 1,$$

$(3m^2+1)/4$ and $3m$ are relatively prime and $\nu_a = 3m$. Thus, by Theorem 3.1, we have constructed a 2-cell imbedding of $K_{3(m)}$ in S_k, with 2 regions of length $3m^2$. This completes the proof, but we note that $k = [3m(m-1)]/2$ is also established by Theorem 3.2.

COROLLARY 4.1. *For m odd, $K_{3(m)}$ has a 2-cell imbedding in S_k if and only if*

$$\binom{m-1}{2} \leqslant k \leqslant 3\binom{m}{2}.$$

By Theorem 2.3 or Theorem 2.4, and Theorem 5.1 of the next section, we see that the restriction "m odd" can be dropped from the statement of Corollary 4.1.

5. Maximum genus. In this section we show that all complete n-partite graphs are upper-imbeddable. The existence proof is by double induction. Although the proof may be reinterpreted constructively, it does not provide a precise algebraic blueprint for the construction, as does a reduced constellation.

The following lemma from [11] will be helpful. $V(G)$ denotes the vertex set of the graph G.

LEMMA 5.1. *Let G be a connected graph having a 2-cell imbedding with i regions, $i = 1, 2$. Let $S \subseteq V(G)$ be a non-empty set of order k. If $i = 2$, assume that S contains two vertices s and t, with s in one region and t in the* *other. Let u be a vertex not in $V(G)$, and define the graph G_k by the formula (where $+\sum$ means adding the indicated edges to the graph)*

$$G_k = G+\sum[u, s], \quad s\in S,$$

where $V(G_k) = V(G)\cup\{u\}$. Then G_k has a 2-cell imbedding with i regions if k is odd, and one with $3-i$ regions if k is even.

THEOREM 5.1. *$K_{m_1,m_2,\ldots,m_n}$ is upper-imbeddable.*

Proof. We use induction on n. By Theorem 2.10, the induction is anchored at $n = 2$.

Now suppose $K_{m_1,m_2,\ldots,m_k}$ is upper-imbeddable; we use induction on m_{k+1} to show that $K_{m_1,m_2,\ldots,m_k,m_{k+1}}$ is upper-imbeddable.

If $m_{k+1} = 1$, we let S in Lemma 5.1 be $V(K_{m_1,m_2,\ldots,m_k})$, so that $K_{m_1,m_2,\ldots,m_k,1}$ is upper-imbeddable.

Now suppose $K_{m_1,m_2,\ldots,m_k,t-1}$ is upper-imbeddable and again let $S = V(K_{m_1,m_2,\ldots,m_k})$. If $\beta(K_{m_1,m_2,\ldots,m_k,t-1})$ is odd, we must find vertices s and v of $V(K_{m_1,m_2,\ldots,m_k})$ which are in different regions of the two-region imbedding of $K_{m_1,m_2,\ldots,m_k,t-1}$. Since each edge in $K_{m_1,m_2,\ldots,m_k,t-1}$ involves a vertex of $K_{m_1,m_2,\ldots,m_k}$, vertices s and v clearly exist. By Lemma 5.1, then, regardless of the parity of $\beta(K_{m_1,m_2,\ldots,m_k,t-1})$, we may add edges from a new vertex w to each of the vertices in $K_{m_1,m_2,\ldots,m_k}$ so that the resultant graph $K_{m_1,m_2,\ldots,m_k,t}$ has a 2-cell imbedding with one or two regions. Thus $K_{m_1,m_2,\ldots,m_k,m_{k+1}}$ and, in turn, $K_{m_1,m_2,\ldots,m_n}$ are upper-imbeddable.

6. Conclusion. Due to Theorem 5.1, the remaining questions of interest in connection with 2-cell imbeddings of complete n-partite graphs appear to be the following:

(i) For $G = K_{m_1,m_2,\ldots,m_n}$ and $n\geqslant 3$, find $\gamma(G)$ in the vast majority of cases where it remains to be determined. (**P 992**)

(ii) Further, study self-dual imbeddings of these graphs.

In both of the above, it would seem natural to first focus on the case $G = K_{n(m)}$, so that the theory of reduced constellations could apply. Finally,

(iii) Find reduced constellations for $\gamma_M(K_{n(m)})$. (**P 993**)

REFERENCES

[1] H. S. M. Coxeter and W. O. Moser, *Generators and relations for discrete groups*, Berlin 1957.

[2] R. A. Duke, *The genus, regional number, and Betti number of a graph*, Canadian Journal of Mathematics 18 (1966), p. 817-822.

[3] J. Edmonds, *A combinatorial representation for polyhedral surfaces*, Notices of the American Mathematical Society 7 (1960), p. 646.

[4] B. Garman, *On the genus of $K_{n,n,n,n}$*, ibidem 20 (1973), A-564.

[5] J. L. Gross and S. R. Alpert, *Branched coverings of current graph imbeddings*, Bulletin of the American Mathematical Society 79 (1973), p. 942-945.
[6] W. Gustin, *Orientable imbedding of Cayley graphs*, ibidem 69 (1963), p. 272-275.
[7] A. Jacques, *Constellations et propriétés algébriques des graphes topologiques*, Ph. D. Thesis, University of Paris, 1969.
[8] M. Jungerman, *The genus of the symmetric quadripartite graph* (to appear).
[9] D. König, *Theorie der endlichen und unendlichen Graphen*, Leipzig 1936. (Reprinted by Chelsea, New York 1950.)
[10] E. A. Nordhaus, B. M. Stewart and A. T. White, *On the maximum genus of a graph*, Journal of Combinatorial Theory B 11 (1972), p. 258-267.
[11] R. D. Ringeisen, *Determining all compact orientable 2-manifolds upon which $K_{m,n}$ has 2-cell imbeddings*, ibidem 12 (1972), p. 101-104.
[12] G. Ringel, *Das Geschlecht des vollständigen paaren Graphen*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 28 (1965), p. 139-150.
[13] — *Genus of graphs* in *Combinatorial structures and their applications*, New York 1970, p. 361-366.
[14] — and J. W. T. Youngs, *Solution of the Heawood map-coloring problem*, Proceedings of the National Academy of Sciences of the U.S.A. 60 (1968), p. 438-445.
[15] — *Das Geschlecht des symmetrischen vollständigen dreifarbigen Graphen*, Commentarii Mathematici Helvetici 45 (1970), p. 152-158.
[16] A. T. White, *The genus of Cartesian products of graphs*, Ph. D. Thesis, Michigan State University, 1969.
[17] — *The genus of the complete tripartite graph $K_{mn,n,n}$*, Journal of Combinatorial Theory 7 (1969), p. 283-285.
[18] — *Graphs, groups and surfaces*, Amsterdam 1973.
[19] — *Orientable imbeddings of Cayley graphs*, Duke Mathematical Journal 41 (1974), p. 353-371.
[20] J. W. T. Youngs, *Minimal imbeddings and the genus of a graph*, Journal of Mathematics and Mechanics 12 (1963), p. 303-315.
[21] — *The Heawood map-coloring conjecture* in *Graph theory and theoretical physics*, London 1967, p. 313-354.

STATE UNIVERSITY OF NEW YORK
BINGHAMTON, NEW YORK

COLGATE UNIVERSITY
HAMILTON, NEW YORK

WESTERN MICHIGAN UNIVERSITY
KALAMAZOO, MICHIGAN

Reçu par la Rédaction le 6. 2. 1975