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ON INTERPOLATION POLYNOMIALS OF THE HERMITE-FEJÉR TYPE

BY

T. M. MILLS (BENDIGO, VICTORIA)

1. If $f(x)$ is defined on the interval $[-1,1]$, then we define the *Hermite-Fejér interpolating polynomial* to be the polynomial of degree $2n-1$ uniquely determined by the conditions

$$H_n[f, x_{kn}] = f(x_{kn}), \quad k = 1, 2, \ldots, n,$$
$$H'_n[f, x_{kn}] = 0, \quad k = 1, 2, \ldots, n,$$

where

$$x_{kn} = \cos\frac{2k-1}{2n}\pi, \quad k = 1, 2, \ldots, n. \tag{1}$$

It was shown by Fejér [2] that if $f(x)$ is continuous on $[-1,1]$, then $H_n[f,x]$ converges to $f(x)$ uniformly on $[-1,1]$ as n tends to infinity.

More recently Stancu [4] considered the polynomial $R_n[f,x]$ of degree $4n-1$ uniquely determined by the conditions

$$\begin{aligned} R_n[f, x_{kn}] &= f(x_{kn}), \quad k = 1, 2, \ldots, n, \\ R_n^{(j)}[f, x_{kn}] &= 0, \quad k = 1, 2, \ldots, n, \end{aligned} \tag{2}$$

for $j = 1, 2, 3$. Again, the nodes x_{kn} are defined by (1). Stancu showed that if $f(x)$ is continuous on $[-1,1]$, then $R_n[f,x]$ converges uniformly on $[-1,1]$ to $f(x)$ as n tends to infinity. Furthermore he proved that

$$\|R_n[f,x]-f(x)\| = O(1)w(f; n^{-1/2}),$$

where the $O(1)$ is independent of f and n, $w(f;\delta)$ is the modulus of continuity of f, and $\|\cdot\|$ denotes the uniform norm on $[-1,1]$.

Later Florica [3] improved this estimate by showing that

$$\|R_n[f,x]-f(x)\| = O(1)w\left(f; \frac{\log n}{n}\right) \quad \text{for } n > 1.$$

In this paper we shall improve this last estimate and show, by obtaining a lower bound as well, that the estimate is, in some sense, best possible.

Let $w(\delta)$ be a modulus of continuity and let $C(w)$ be the class of functions f which satisfy

$$|f(x)-f(y)| \leqslant w(|x-y|) \quad \text{for all } x \text{ and } y \text{ in } [-1,1].$$

Then we prove the following result:

THEOREM. *There are positive constants A and B such that*

$$\frac{A}{n}\sum_{r=2}^{n} w\left(\frac{1}{r}\right) \leqslant \sup_{f\in C(w)} \|R_n[f,x]-f(x)\| \leqslant \frac{B}{n}\sum_{r=1}^{n} w\left(\frac{1}{r}\right) \quad \textit{for all } n>1.$$

It should be noted that Bojanic [1] has obtained similar estimates for Fejér's operator. Also, this estimate improves that of Florica since

$$\frac{1}{n}\sum_{r=1}^{n} w\left(\frac{1}{r}\right) = \frac{1}{n}\sum_{r=1}^{n} w\left(\frac{n\log n}{rn\log n}\right)$$
$$\leqslant \frac{1}{n}\sum_{r=1}^{n}\left(1+\frac{n}{r\log n}\right) w\left(\frac{\log n}{n}\right) = O(1)\, w\left(\frac{\log n}{n}\right).$$

2. First, let us find an upper bound for the error. Stancu has shown that

$$R_n[f,x] = \sum_{k=1}^{n} f(x_k)s_k(x),$$

where

$$s_k(x) = F_k(x)+G_k(x)+H_k(x),$$

$$F_k(x) = \frac{1}{n^4}(1-x_k^2)(1-x^2)\left(\frac{T_n(x)}{x-x_k}\right)^4, \tag{3}$$

$$G_k(x) = \frac{1}{6n^4}(x-x_k)^2(4n^2-1)(1-xx_k)\left(\frac{T_n(x)}{x-x_k}\right)^4, \tag{4}$$

$$H_k(x) = \frac{1}{2n^4}\left(\frac{T_n^2(x)}{x-x_k}\right)^2, \tag{5}$$

and $x_k = x_{kn}$ are, as in (1), the zeros of the Chebyshev polynomial $T_n(x) = \cos(n\arccos x)$ of the first kind.

Let $x\in[-1,1]$ and let j be an integer such that

$$|x-x_j| \leqslant |x-x_k| \quad \text{for } k = 1,2,\ldots,n.$$

Now, since $R_n[f,x]$ is uniquely determined by (2), it follows that

$$R_n[1,x] = \sum_{k=1}^{n} s_k(x) = 1.$$

Hence

$$
\begin{aligned}
&|R_n[f, x]-f(x)| \\
&= \Big|\sum_{k=1}^{n} \big(f(x_k)-f(x)\big)s_k(x)\Big| \leqslant \sum_{k=1}^{n} |f(x_k)-f(x)|\, s_k(x) \\
&= \sum_{k=1}^{j-1} |f(x_k)-f(x)|\, s_k(x) + |f(x_j)-f(x)|\, s_j(x) + \sum_{k=j+1}^{n} |f(x_k)-f(x)|\, s_k(x) \\
&= W_1+W_2+W_3.
\end{aligned}
$$

If $j = 1$ or n, then either W_1 or W_2 will not appear. We begin by estimating W_1.

Suppose then that $k = j-i$, where $1 \leqslant i \leqslant j-1$, $x = \cos\theta$, and $x_k = \cos\theta_k$ as in (1). It then follows that, for $f \in C(w)$,

$$
\begin{aligned}
(6) \qquad |f(x_k)-f(x)| &\leqslant w(|x-x_k|) = w(|\cos\theta-\cos\theta_k|) \\
&\leqslant w(|\theta-\theta_k|) = O(1)w\left(\frac{i}{n}\right).
\end{aligned}
$$

To estimate $s_k(x)$ we estimate each of $F_k(x)$, $G_k(x)$, and $H_k(x)$ in turn. From (3) we may deduce that

$$
(7) \qquad F_k(x) = \frac{1}{16n^4}\, \frac{\sin^2\theta_k}{\sin^2\big((\theta+\theta_k)/2\big)}\, \frac{\sin^2\theta}{\sin^2\big((\theta+\theta_k)/2\big)}\, \frac{T_n^4(x)}{\sin^4\big((\theta-\theta_k)/2\big)}.
$$

But,

$$
\sin\theta_k \leqslant \sin\theta_k+\sin\theta \leqslant 2\sin\frac{\theta+\theta_k}{2}
$$

and, similarly,

$$
\sin\theta \leqslant 2\sin\frac{\theta+\theta_k}{2}.
$$

Also, $|T_n(x)| = |\cos n\theta| \leqslant 1$ and, by Jordan's inequality,

$$
\left(\sin^4\frac{\theta-\theta_k}{2}\right)^{-1} = O(1)(\theta-\theta_k)^{-4} = O(1)(n^4 i^{-4}).
$$

If we use all these inequalities in (7), we obtain

$$
(8) \qquad F_k(x) = O(1)i^{-4}.
$$

Now let us consider $G_k(x)$. From (4) and [1], p. 72, we deduce that

$$
(9) \qquad G_k(x) = O(1)n^{-2}(1-xx_k)T_n^2(x)(x-x_k)^{-2} = O(1)i^{-2}.
$$

From (5) we obtain

$$H_k(x) = O(1)n^{-4}T_n^2(x)(x-x_k)^{-2} = O(1)n^{-4}(x-x_k)^{-2}$$
$$= O(1)n^{-4}\left(\sin\frac{\theta+\theta_k}{2}\sin\frac{\theta-\theta_k}{2}\right)^{-2}$$
$$= O(1)n^{-4}\left(\sin\frac{\theta-\theta_k}{2}\right)^{-4} = O(1)n^{-4}\left(\frac{i}{n}\right)^{-4}.$$

Hence

$$H_k(x) = O(1)i^{-4}. \tag{10}$$

Finally, from (8), (9) and (10) it follows that

$$s_k(x) = O(1)i^{-2}, \quad \text{where } k = j-i \text{ and } i \geqslant 1. \tag{11}$$

By (6) and (11) we now have

$$W_1 = O(1)\sum_{i=1}^{j-1} w\left(\frac{i}{n}\right)i^{-2} = O(1)\sum_{i=1}^{n} w\left(\frac{i}{n}\right)i^{-2} = O(1)\sum_{i=1}^{n} w\left(\frac{1}{i}\right)n^{-1}. \tag{12}$$

In this last step, we have used a result which is implicit in the paper by Bojanic [1].

To estimate $W_2 = |f(x_j)-f(x)|s_j(x)$ we note that

$$0 \leqslant s_j(x) \leqslant \sum_{k=1}^{n} s_k(x) = 1$$

and

$$|f(x_j)-f(x)| \leqslant w(|\cos\theta - \cos\theta_j|) \leqslant w(|\theta-\theta_j|) = O(1)w\left(\frac{1}{n}\right).$$

Consequently,

$$W_2 = O(1)w\left(\frac{1}{n}\right). \tag{13}$$

To estimate W_3 we proceed in the manner used to estimate W_1 and, thereby, we obtain

$$W_3 = O(1)\sum_{r=1}^{n}\frac{w(1/r)}{n}. \tag{14}$$

The upper estimate stated in the theorem now follows from (12), (13), and (14).

3. Let us now turn to the lower bound stated in the theorem. Let $g(x) = w(|x-\frac{1}{2}|)$. From the properties of a modulus of continuity it is easy to see that $g(x)\in C(w)$. Furthermore,

$$(15)\qquad \sup_{f\in C(w)} \|R_n[f,x]-f(x)\| \geqslant \|R_n[g,x]-g(x)\|$$

$$\geqslant \left|R_n\left[g,\frac{1}{2}\right]-g\left(\frac{1}{2}\right)\right| = R_n\left[g,\frac{1}{2}\right]$$

$$= \sum_{k=1}^{n} w\left(\left|x_k-\frac{1}{2}\right|\right) s_k\left(\frac{1}{2}\right) \geqslant \sum_{k=1}^{n} w\left(\left|x_k-\frac{1}{2}\right|\right) G_k\left(\frac{1}{2}\right).$$

Now

$$G_k\left(\frac{1}{2}\right) = \frac{4n^2-1}{6n^4}\left(1-\frac{x_k}{2}\right)\left(\frac{T_n^2(1/2)}{1/2-x_n}\right)^2$$

$$\geqslant \frac{4n^2-1}{24n^4}\left(1-\frac{x_k}{2}\right)\left(\frac{T_n(1/2)}{1/2-x_k}\right)^2 \geqslant \frac{1}{8n^2}\left(1-\frac{x_k}{2}\right)\left(\frac{T_n(1/2)}{1/2-x_k}\right)^2$$

since $T_n^2(1/2) = \cos^2(n\pi/3) \geqslant 1/4$.

Therefore, we have

$$(16)\qquad \sum_{k=1}^{n} w\left(\left|x_k-\frac{1}{2}\right|\right) G_k\left(\frac{1}{2}\right) \geqslant \frac{1}{8n^2}\sum_{k=1}^{n} w\left(\left|x_k-\frac{1}{2}\right|\right)\left(1-\frac{x_k}{2}\right)\left(\frac{T_n(1/2)}{1/2-x_k}\right)^2$$

$$\geqslant c\sum_{r=2}^{n}\frac{w(1/r)}{n} \qquad \text{for } n\geqslant 2;$$

this last inequality follows from Bojanic's work [1], p. 74 and 75.

The lower estimate stated in the theorem now follows immediately from (15) and (16). This completes the proof of the theorem.

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DEPARTMENT OF MATHEMATICS
BENDIGO COLLEGE OF ADVANCED EDUCATION
BENDIGO, VICTORIA
AUSTRALIA

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