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A NOTE ON THE SEMI-INVERSE APPROACH TO OSCILLATION THEORY FOR ORDINARY SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. Introductory remarks. The semi-inverse technique was introduced by Saint Venant to deal with the problem of finding the stress function which would solve a specific boundary value problem of elasticity. Since the original problem appeared insolvable, Saint Venant proposed the following approach:

Choosing a specific form of a stress function, he worked back to discover to what particular boundary value problem this stress function corresponded.

By experimenting with various stress functions and using the superposition principle, the engineers were able to solve important problems of linear elasticity. This note points out the fact that this technique is readily available to experiment with a much simpler problem of oscillatory behavior of solutions of linear second-order ordinary differential equations.

It is well known that a Kummer type transformation is available, which transforms any equation of the form

$$(ay')' + cy = 0, \quad a \in C^2,\ c \in C,\ a > 0, \tag{1}$$

into an equation

$$x'' + \sigma(t)x = 0, \quad t \in [t_0, \infty), \tag{2}$$

preserving the oscillatory properties of the solutions.

Hence, it is sufficient to consider only the equations of form (2). (See, for example, an expository article of Willett [2] for the details.)

It was known to Kamke, and was rediscovered at regular intervals later on, that the behavior of solutions of equation (2) is closely connected with the properties of solutions of the equation

$$\varrho'' + \sigma(t)\varrho - \varrho^{-3} = 0, \quad \varrho(t) > 0,\ \varrho(t) \in C^2,\ t \in [t_0, \infty). \tag{3}$$

In particular, the solutions of (2) are oscillatory if and only if the solutions of (3) have the property

$$\int_{t_0}^{\infty} [\varrho(\xi)]^{-2} d\xi = +\infty.$$

(For justification of this statement see, for example, either Willett [2] or Ráb [1].)

2. Explanation of the procedure. Let us choose an arbitrary function $\varrho(t) \in C^2[t_0, \infty)$, $\varrho(t) > 0$ on $[t, \infty)$, and satisfying either

(4a) $$\int_{t_0}^{\infty} [\varrho(\xi)]^{-2} d\xi = \infty$$

or

(4b) $$\int_{t_0}^{\infty} [\varrho(\xi)]^{-2} d\xi < \infty.$$

Writing $u(t) = \varrho'/\varrho$, we compute the function $-u' - u^2 + \varrho^{-4} = \sigma(t)$. Then, with this choice of $\sigma(t)$, equation (2) is oscillatory if (4a) is satisfied, and non-oscillatory if (4b) is satisfied. (This statement is known to be true, moreover, the proof of it follows directly from our previous remarks.) Alternately, we could observe that $x(t)$ is a solution of (2) if and only if $w(t) = \varrho^{-1}(t)x(t)$ is a solution of the equation

(5) $$\left(\varrho^2 w'(t)\right)' + \varrho^{-2} w(t) = 0,$$

which has a general solution

$$w(t) = A\cos\left(B + \int_{t_0}^{t} \varrho^{-2}(s)\,ds\right).$$

Clearly, $w(t)$ and $x(t)$ must have an identical oscillatory behavior.

3. Results of some experiments.

Example 1. *The equation*

$$x'' + [t^{-4\lambda} + \lambda(1-\lambda)t^{-2}]x = 0, \quad t \geqslant 1,$$

is oscillatory if and only if $\lambda \leqslant 1/2$. (This is, of course, easily deduced from the criterion of Kneser and Hille.)

Proof. We choose $\varrho = t^{\lambda}$. Then $u = \lambda t^{-1}$, and we have $-u' - u^2 = \lambda t^{-2} - \lambda^2 t^{-2}$ and $\sigma(t) = \lambda(1-\lambda)t^{-2} + t^{-4\lambda}$. Since

$$\int_1^{\infty} \varrho^{-2}(\xi)\,d\xi = \int_1^{\infty} t^{-2\lambda} dt \begin{cases} < \infty & \text{if } \lambda > 1/2, \\ = \infty & \text{if } \lambda \leqslant 1/2, \end{cases}$$

our claim is confirmed.

Example 2. *The equation*

$$x'' + \left(\frac{1}{4} + K(\log t)^{-4}\right) t^{-2} x = 0, \qquad t \geqslant 2,$$

is non-oscillatory for any constant $K > 0$.

Proof. Choose $\varrho = at^{\lambda}\log t$, where a is any non-zero constant. Then we have

$$u = \frac{\varrho'}{\varrho} = \lambda t^{-1} + (t\log t)^{-1},$$

$$-u' = \lambda t^{-2} + t^{-2}(\log t)^{-1} + t^{-2}(\log t)^{-2},$$

$$\begin{aligned}\sigma(t) &= -u' - u^2 + \varrho^{-4} = \lambda(1-\lambda)t^{-2} + t^{-2}(\log t)^{-1} \\ &\quad -2\lambda t^{-2}(\log t)^{-1} + a^4 t^{-4\lambda}(\log t)^{-4} \\ &= t^{-2}[\lambda(1-\lambda) + (1-2\lambda)(\log t)^{-1}] + a^4 t^{-4\lambda}(\log t)^{-4}.\end{aligned}$$

Putting $\lambda = 1/2$, we obtain

$$\sigma(t) = \frac{1}{4} t^{-2} + a^4 t^{-2}(\log t)^{-4}.$$

Since we have

$$\frac{1}{a^2}\int_2^{\infty} \varrho^{-2}(\xi)\,d\xi = \int_2^{\infty} t^{-1}(\log t)^{-2}\,dt = -(\log t)^{-1}\Big|_2^{\infty} < \infty,$$

the claim is proved.

Example 3. *The equation*

$$x'' + \left[\frac{\sin t}{K + \sin t} + (K + \sin t)^{-4}\right] x = 0, \qquad t \geqslant 0,$$

is oscillatory for any $K > 1$.

Proof. Choose $\varrho = K + \sin t$ and check that

$$\int_2^{\infty} (K + \sin t)^{-2}\,dt = \infty.$$

Other examples are easily constructed including the following result: *Let* $c(t)$ *be of the form*

$$c(t) = -\varphi'' - (\varphi')^2 + \exp(-4\varphi) \quad \textit{for some } \varphi \in C^2[t_0, \infty].$$

Then $x'' + c(t)x = 0$, $t \geqslant T$, *is oscillatory if and only if*

$$\int_T^{\infty} \exp(-2\varphi)\,dt = \infty.$$

Proof. Choose $\varrho = \exp(\varphi(t))$.

REFERENCES

[1] M. Ráb, *Kriterien für die Oszillation der Lösungen der Differentialgleichung* $[p(x)y']' + g(x)y = 0$, Časopis pro Pěstování Matematiky 84 (1959), p. 335-370 (Erratum, ibidem 85 (1960), p. 91).

[2] D. Willett, *Classification of second-order ordinary differential equations with respect to oscillation*, Advances in Mathematics 3 (1969), p. 594-623.

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