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ON THE NON-CONVERGENCE OF SUCCESSIVE APPROXIMATIONS IN THE DARBOUX PROBLEM FOR THE EQUATION $z''_{xy} = f(x, y, z)$

BY

M. KISIELEWICZ (ZIELONA GÓRA)

1. Introduction. Many authors ([2]-[5], [8], [9], [11]) have discussed the problem of the convergence of successive approximations to a solution of differential equations when the latter is unique in virtue of a uniqueness criterion. A well-known example due to Müller ([10], see also [4], p. 53) shows that the continuity of the right-hand side of a given equation and uniqueness of its solutions are not sufficient to guarantee the convergence of its successive approximations.

It was proved in [1] that non-uniqueness of a solution of the Darboux problem

$$(1)\qquad \begin{cases} z''_{xy} = f(x, y, z) & \text{for } (x, y) \in [0, a]\times[0, b], \\ z(x, 0) = \sigma(x) & \text{for } x \in [0, a], \\ z(0, y) = \tau(y) & \text{for } y \in [0, b] \end{cases}$$

is a rather rare case. In this paper we show that such is also non-convergence of successive approximations of (1).

Let R denote the real line, and let R^m be an m-dimensional linear vector space with the norm

$$\|x\| = \max(|x_1|, |x_2|, \ldots, |x_m|), \quad \text{where } x = (x_1, x_2, \ldots, x_m).$$

Let P denote the set in R^{m+2} defined by

$$P = \{(x, y, z): (x, y) \in D, z \in R^m\}, \quad \text{where } D = [0, a]\times[0, b].$$

In this paper we consider functions $f: P \to R^m$ satisfying the following hypotheses:

(C_1) $f(\cdot, \cdot, z): D \to R^m$ is measurable for all $z \in R^m$.

(C_2) $f(x, y, \cdot): R^m \to R^m$ is continuous for a.e. (1) $(x, y) \in D$.

(1) a.e. stands for *almost every*.

(C_3) There exists a number $M > 0$ such that $\|f(x, y, z)\| \leqslant M$ for a.e. $(x, y) \in D$ and for all $z \in R^m$.

2. Fundamental metric space and basic theorems. Let Q_a be a set defined by

$$Q_a = \{(x, y, z) \in P: \|z\| \leqslant a\} \quad \text{for } a > 0.$$

In the proof of the main theorem of this paper we shall use the following results:

THEOREM 1 (Alexiewicz-Orlicz). *Let $f: Q_a \to R^m$ satisfy (C_1) - (C_3) for $a > 0$. Then there exists a sequence $\{f_n\}$ of continuous functions from Q_a to R^m such that*

(i) $\|f_n(x, y, z)\| \leqslant M$ *for* $(x, y, z) \in Q_a$ *and* $n = 1, 2, \ldots$,

(ii) $\lim\limits_{n\to\infty} \max\limits_{\|z\|\leqslant a} \|f_n(x, y, z) - f(x, y, z)\| = 0$ *for a.e.* $(x, y) \in D$.

In a way similar to that in [6] we can easily obtain

THEOREM 2. *Suppose that $f: Q_a \to R^m$ satisfies (C_1)-(C_3) for $a > 0$. Then for every $\varepsilon > 0$ there exists a function $f^\varepsilon: Q_a \to R^m$ such that*

(i) $\|f^\varepsilon(x, y, z)\| \leqslant M$ *for* $(x, y, z) \in Q_a$,

(ii) $\max\limits_{\|z\|\leqslant a} \|f^\varepsilon(x, y, z) - f(x, y, z)\| \leqslant \varepsilon$ *for a.e.* $(x, y) \in D$,

(iii) $f^\varepsilon(x, y, z)$ *has continuous partial derivatives of all orders with respect to* $z_1, z_2, \ldots, z_n$, *where* $z = (z_1, z_2, \ldots, z_n)$.

Let us denote by $F(P)$ the set of all functions $f: P \to R^m$ satisfying (C_1)-(C_3). We can define in $F(P)$ an equivalence relation $\sim$ in the following way: for any $f_1, f_2 \in F(P)$ we write $f_1 \sim f_2$ if there is a set $A \subset D$ of measure zero such that

$$f_1(x, y, z) = f_2(x, y, z) \quad \text{for every } z \in R^m \text{ and } (x, y) \in D \setminus A.$$

Let $\mathscr{F}(P) = F(P)/\sim$. For given $f \in F(P)$ we shall denote by $\tilde{f}$ the element of $\mathscr{F}(P)$ which contains f. We consider $\mathscr{F}(P)$ together with a metric function ϱ defined by

$$\varrho(\tilde{f}_1, \tilde{f}_2) = \|\tilde{f}_1 - \tilde{f}_2\|_{\mathscr{F}},$$

where

$$\|\tilde{f}\|_{\mathscr{F}} = \iint\limits_D \sup_z \{\|f(x, y, z)\| : (x, y, z) \in P\}\, dx\, dy$$

for $\tilde{f}, \tilde{f}_1, \tilde{f}_2 \in \mathscr{F}(P)$ and $f \in \tilde{f}$.

LEMMA 1. *$(\mathscr{F}(P), \varrho)$ is a complete metric space.*

Proof. Let $\{\tilde{f}_n\}$ be a sequence of $\mathscr{F}(P)$ such that

$$\|\tilde{f}_n - \tilde{f}_m\|_{\mathscr{F}} \to 0 \quad \text{as } n, m \to \infty,$$

and let $f_n \in \tilde{f}_n, f_m \in \tilde{f}_m$. For every $\varepsilon > 0$ there is a number $N = N(\varepsilon)$ such that

$$\iint\limits_{D} \sup_{z} \{\|f_n(x, y, z) - f_m(x, y, z)\| : (x, y, z) \in P\} \, dx\, dy \leqslant \varepsilon \qquad \text{for } n, m \geqslant N(\varepsilon).$$

Suppose that $\{n_k\}$ is such that $n_1 < n_2 < \dots$ and $n_{k-1} \geqslant N(1/2^{2k})$. Then

$$\iint\limits_{D} \sup_{z} \{\|f_{n_k}(x, y, z) - f_{n_{k-1}}(x, y, z)\| : (x, y, z) \in P\} \, dx\, dy \leqslant 1/2^{2k} \qquad \text{for } k = 1, 2, \dots$$

Taking

$$A_k = \{(x, y) \in D : \sup_{z} [\|f_{n_k}(x, y, z) - f_{n_{k-1}}(x, y, z)\| : (x, y, z) \in P] > 1/2^k\},$$

we have

$$1/2^{2k} \geqslant \iint\limits_{A_k} \sup_{z} \{\|f_{n_k}(x, y, z) - f_{n_{k-1}}(x, y, z)\| : (x, y, z) \in P\} \, dx\, dy \geqslant (1/2^k)\mu(A_k).$$

Then $\mu(A_k) \leqslant 1/2^k$. Let

$$A = \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} A_k.$$

Since

$$\mu(A) \leqslant \mu\Big(\bigcup_{k=i}^{\infty} A_k\Big) \leqslant \sum_{k=i}^{\infty} \mu(A_k) < \sum_{k=i}^{\infty} 1/2^k = 1/2^{i-1} \qquad \text{for } i = 1, 2, \dots,$$

we have $\mu(A) = 0$. Let $A^{\sim} = D \setminus A$ and $A_k^{\sim} = D \setminus A_k$. We have

$$A^{\sim} = \bigcup_{i=1}^{\infty} \bigcap_{k=1}^{\infty} A_k^{\sim}.$$

Then $(x, y) \in A^{\sim}$ implies the existence of a number i such that, for each $k \geqslant i$, we have

$$\sup_{z} \{\|f_{n_k}(x, y, z) - f_{n_{k-1}}(x, y, z)\| : (x, y, z) \in P\} \leqslant 1/2^k.$$

Therefore

$$\sum_{k=i}^{\infty} \sup_{z} \{\|f_{n_k}(x, y, z) - f_{n_{k-1}}(x, y, z)\| : (x, y, z)\} < \infty \qquad \text{for } (x, y) \in A^{\sim}.$$

Then the series

$$f_{n_0}(x, y, z) + \sum_{k=1}^{\infty} [f_{n_k}(x, y, z) - f_{n_{k-1}}(x, y, z)]$$

is absolutely and uniformly convergent on $A^{\sim}$ independently of $z \in R^m$. Let $f: P \to R^m$ be defined by

$$f(x, y, z) = \begin{cases} \lim\limits_{k\to\infty} f_{n_k}(x, y, z) & \text{for } (x, y) \in A^{\sim} \text{ and } z \in R^m, \\ 0 & \text{for } (x, y) \in A \text{ and } z \in R^m. \end{cases}$$

The function f satisfies (C_1)-(C_3). We shall show that

$$\|\tilde{f}_n - \tilde{f}\| \to 0 \quad \text{as } n \to \infty.$$

For $n, k \geqslant N(\varepsilon)$ we have

$$\iint\limits_D \sup_z \{\|f_n(x, y, z) - f_{n_k}(x, y, z)\| : (x, y, z) \in P\}\, dx\, dy \leqslant \varepsilon.$$

Taking, for fixed n,

$$\Psi_k(x, y) = \sup_z \{\|f_n(x, y, z) - f_{n_k}(x, y, z)\| : (x, y, z) \in P\},$$

we have

$$\|\tilde{f}_n - \tilde{f}\|_{\mathscr{F}} = \iint\limits_D \lim_{k\to\infty} \Psi_k(x, y)\, dx\, dy.$$

In virtue of Fatou's Lemma we obtain

$$\iint\limits_D \varliminf_{k\to\infty} \Psi_k(x, y)\, dx dy \leqslant \varliminf_{k\to\infty} \iint\limits_D \Psi_k(x, y)\, dx\, dy = \varliminf_{k\to\infty} \|\tilde{f}_n - \tilde{f}_{n_k}\|_{\mathscr{F}} \leqslant \varepsilon \quad \text{for } n \geqslant N(\varepsilon).$$

This completes the proof.

Let $\sigma \in \mathrm{AC}([0, a], R^m)$ and $\tau \in \mathrm{AC}([0, b], R^m)$, where $\mathrm{AC}(I, R^m)$ denotes the set of all absolutely continuous functions from I to R^m. It was proved in [1] and [7] that for every $\tilde{f} \in \mathscr{F}(P)$ there exists at least one solution of (1). Obviously, we have to assume that $\sigma(0) = \tau(0)$.

For $\tilde{f} \in \mathscr{F}(P)$ and for fixed σ and τ defined as above we consider the sequence $\{z_n^f\}$ of successive approximations defined by

$$z_{n+1}^f(x, y) = \varphi_0(x, y) + \int\limits_0^x \int\limits_0^y f\big(\xi, \eta, z_n^f(\xi, \eta)\big)\, d\xi d\eta \quad \text{for } n = 0, 1, \ldots, \tag{2}$$

where $z_0^f \in \mathrm{C}(D)$ and $\varphi_0(x, y) = \sigma(x) + \tau(y) - \sigma(0)$.

We call the sequence $\{z_n^f\}$ *converging in* D if $\{z_n^f(x, y)\}$ converges for every $(x, y) \in D$. It is easy to verify, as in the theory of ordinary differential equations, that if $f \in \mathscr{F}(P)$ satisfies the Lipschitz condition with respect to z, then $\{z_n^f\}$ is uniformly convergent to the unique solution of (1). It is easy to see that

$$\max_D \|z_n^f(x, y)\| \leqslant a \quad \text{for each } n = 0, 1, \ldots,$$

where

$$a = \max\{\sup_D \|z_0^f(x, y)\|, \sup_D \|\varphi_0(x, y)\| + abM\}.$$

3. Non-convergence of successive approximations. Now we show that non-convergence of the successive approximations (2) is a rather rare case. More precisely, the set $\mathscr{A} \subset \mathscr{F}(P)$ of those functions $\tilde{f}$ for which $\{z_n^f\}$ is not convergent is of Baire's first category in the space $(\mathscr{F}(P), \varrho)$.

For given $\tilde{f} \in \mathscr{F}(P)$ and $(x, y) \in D$ let $\Delta(\tilde{f}, x, y)$ be defined by

$$\Delta(\tilde{f}, x, y) = \limsup_{n\to\infty} \{\operatorname{diam} E[z_n^f(x, y)]\},$$

where

$$E[z_n^f(x, y)] = \{z_n^f(x, y), z_{n+1}^f(x, y), \ldots\},$$

and $\operatorname{diam} A$ denotes the diameter of a set $A \subset R^m$. Obviously, the equality $\Delta(\tilde{f}, x, y) = 0$ for each $(x, y) \in D$ is equivalent to the convergence of $\{z_n^f\}$. Thus the sequence $\{z_n^f\}$ is not converging in D if and only if there is $(\tilde{x}, \tilde{y}) \in D$ such that $\Delta(\tilde{f}, \tilde{x}, \tilde{y}) > 0$. Let $\{(x_r, y_r)\}$ be a sequence of points of D dense in D and let

$$\Omega_{Mpr} = \{\tilde{f} \in \mathscr{F}(P)\colon \|f\|_{\mathscr{F}} \leqslant M, \Delta(\tilde{f}, x_r, y_r) \geqslant 1/p\}.$$

LEMMA 2. *Ω_{Mpr} are closed subsets of $\mathscr{F}(P)$ for all $M, p, r = 1, 2, \ldots$*

Proof. Suppose that $\{\tilde{f}_k\}$ is a sequence of Ω_{Mpr} such that

$$\|\tilde{f}_k - \tilde{f}\|_{\mathscr{F}} \to 0 \quad \text{as } k\to\infty,$$

where $\tilde{f} \in \mathscr{F}(P)$. It is easy to see that $\|\tilde{f}\|_{\mathscr{F}} \leqslant M$. Furthermore, there exists a subsequence $\{\tilde{f}_{n_k}\}$ of $\{\tilde{f}_k\}$ such that

$$\sup_z \{\|f_{n_k}(x, y, z) - f(x, y, z)\|\colon (x, y, z) \in P\} \to 0 \qquad \text{for a.e. } (x, y) \in D \text{ and } k\to\infty.$$

For each $k = 1, 2, \ldots$ we have $\Delta(\tilde{f}_{n_k}, x_r, y_r) \geqslant 1/p$. Then

$$\sup_{\nu=1,2,\ldots} \{\operatorname{diam} E[z_{n+\nu}^{f_{n_k}}(x_r, y_r)]\} \geqslant 1/p \quad \text{for } n, k = 1, 2, \ldots$$

Hence for every $l = 1, 2, \ldots$ there is ν_l such that

$$\operatorname{diam} E[z_{n+\nu_l}^{f_{n_k}}(x_r, y_r)] \geqslant 1/p - 1/l,$$

i.e.

$$\sup_{(u,v)} \|z_{n+\nu_l+u}^{f_{n_k}}(x_r, y_r) - z_{n+\nu_l+v}^{f_{n_k}}(x_r, y_r)\| > 1/p - 1/l.$$

Then for every $q = 1, 2, \ldots$ there is (u_q, v_q) such that

$$\|z_{n+\nu_l+u_q}^{f_{n_k}}(x_r, y_r) - z_{n+\nu_l+v_q}^{f_{n_k}}(x_r, y_r)\| > 1/p - 1/l - 1/q \quad \text{for } n, k = 1, 2, \ldots$$

Let

$$Z(k, n+\nu_l+u_q)(x, y) = z_{n+\nu_l+u_q}^{f_{n_k}}(x, y).$$

It is easy to see that the family $X \subset C(D, R^m)$ defined by

$$X = \{Z(k, n+\nu_l+u_q)\}_{k,n,l,q=1,2,\ldots}$$

satisfies the hypotheses of Arzela's Theorem. Then there is a subsequence $\{Z(n_k, n+\nu_l+u_q)\}$ of $\{Z(k, n+\nu_l+u_q)\}$ which is uniformly convergent on D. Suppose that

$$\lim_{k\to\infty} Z(n_k, n+\nu_l+u_q) = Z(n+\nu_l+u_q) \quad \text{for every fixed } n, \nu_l \text{ and } u_q.$$

For $n, l, q = 1, 2, \ldots$ and $(x, y) \in D$ we have

$$Z(n+\nu_l+u_q)(x, y) - \sigma(x) - \tau(y) + \sigma(0) - \int_0^x\int_0^y f\big(\xi, \eta, Z(n+\nu_l+u_q-1)(\xi, \eta)\big)d\xi d\eta = \sum_{i=1}^{3} \Lambda_i(x, y),$$

where

$$\Lambda_1(x, y) = Z(n+\nu_l+u_q)(x, y) - Z(n_k, n+\nu_l+u_q)(x, y),$$

$$\Lambda_2(x, y) = \int_0^x\int_0^y \Big[f_{n_k}\big(\xi, \eta, Z(n_k, n+\nu_l+u_q-1)(\xi, \eta)\big) - f_{n_k}\big(\xi, \eta, Z(n+\nu_l+u_q-1)(\xi, \eta)\big)\Big]d\xi\, d\eta,$$

$$\Lambda_3(x, y) = \int_0^x\int_0^y \Big[f_{n_k}\big(\xi, \eta, Z(n+\nu_l+u_q-1)(\xi, \eta)\big) - f\big(\xi, \eta, Z(n+\nu_l+u_q-1)(\xi, \eta)\big)\Big]d\xi\, d\eta.$$

Hence

$$Z(n+\nu_l+u_q)(x, y) = \sigma(x) + \tau(y) - \sigma(0) + \int_0^x\int_0^y f\big(\xi, \eta, Z(n+\nu_l+u_q-1)(\xi, \eta)\big)d\xi\, d\eta$$

for $(x, y) \in D$ and $n, l, q = 1, 2, \ldots$

Therefore $Z(n+\nu_l+u_q) = z^f_{n+\nu_l+u_q}$ for all $n, l, q = 1, 2, \ldots$ Since

$$\|Z(n_k, n+\nu_l+u_q)(x_r, y_r) - Z(n_k, n+\nu_l+v_q)(x_r, y_r)\| \geqslant 1/p - 1/l - 1/q,$$

we have

$$\|z^f_{n+\nu_l+u_q}(x_r, y_r) - z^f_{n+\nu_l+v}(x_r, y_r)\| \geqslant 1/p - 1/l - 1/q \quad \text{for all } n, l, q = 1, 2, \ldots$$

It is not difficult to see that $\Delta(\tilde{f}, x_r, y_r) \geqslant 1/p$. Hence $\tilde{f} \in \Omega_{Mpr}$.

LEMMA 3. *Ω_{Mpr} are non-dense in $\mathscr{F}(P)$ for all $M, p, r = 1, 2, \ldots$*

Proof. Suppose that there are (M, p, r) and a sphere $S_h(\tilde{f}_0) \subset \mathscr{F}(P)$ with center $\tilde{f}_0 \in \mathscr{F}(P)$ and radius $h > 0$ such that $S_h(\tilde{f}_0) \subset \bar{\Omega}_{Mpr}$. By Lemma 2 this means that $S_h(\tilde{f}_0) \subset \Omega_{Mpr}$. Note that for every $\tilde{f} \in \Omega_{Mpr}$ there is a number $a > 0$ such that the sequence $\{z_n^{\tilde{f}}\}$ corresponding to $\tilde{f}$ is the same as $\{z_n^{\tilde{f}|Q}\}$ corresponding to $\tilde{f}|Q$, where $\tilde{f}|Q$ denotes the contraction of $\tilde{f}$ to the set

$$Q = \{(x, y, z) \in P: \|z\| \leqslant a\}.$$

In virtue of Theorem 2, for $\tilde{f}_0|Q$ and $\delta > 0$ there exists a function $f^\delta: Q \to R^m$ such that conditions (i)-(iii) of Theorem 2 are satisfied. Then

$$\max_z \{\|f^\delta(x, y, z) - f_0(x, y, z)\|: (x, y, z) \in Q\} < \delta \quad \text{for a.e. } (x, y) \in D.$$

Taking $\delta < h/(a \cdot b)$ we obtain $\|\tilde{f}^\delta - \tilde{f}_0\|_{\mathscr{F}} < h$. Then

$$\tilde{f}^\delta \in S_h(\tilde{f}_0) \subset \Omega_{Mpr}.$$

But $\tilde{f}^\delta$ is Lipschitz continuous with respect to z, and so

$$\sup_D \Delta(\tilde{f}^\delta, x, y) = 0.$$

Therefore $\tilde{f}^\delta \notin \Omega_{Mpr}$. This completes the proof.

Now we can prove the main result of this paper.

THEOREM 3. *The set $\mathscr{A}$ of those $\tilde{f} \in \mathscr{F}(P)$ for which successive approximations $\{z_n^f\}$ are not converging is of Baire's first category in the space $(\mathscr{F}(P), \varrho)$.*

Proof. In virtue of Lemma 3 it suffices to show that

$$\mathscr{A} = \bigcup_{M=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcup_{r=1}^{\infty} \Omega_{Mpr}.$$

It is easy to see that $\Omega_{Mpr} \subset \mathscr{A}$ for all $M, p, r = 1, 2, \ldots$ Let us observe that

$$\mathscr{A} = \{\tilde{f} \in \mathscr{F}(P);\ \text{there is } (\tilde{x}, \tilde{y}) \in \dot{D}: \Delta(\tilde{f}, \tilde{x}, \tilde{y}) > 0\}.$$

Suppose that $\tilde{f} \in \mathscr{A}$. There exists a positive integer $\bar{p}$ such that $\Delta(\tilde{f}, \tilde{x}, \tilde{y}) \geqslant 2/\bar{p}$. It is not difficult to see that for a given $\bar{p}$ there exists an element $(\bar{x}, \bar{y})$ of the sequence $\{(x_r, y_r)\}$ such that

$$\Delta(\tilde{f}, \bar{x}, \bar{y}) \geqslant \Delta(\tilde{f}, \tilde{x}, \tilde{y}) - 1/\bar{p}.$$

Therefore, there is a positive integer $\bar{r}$ such that

$$\Delta(\tilde{f}, x_{\bar{r}}, y_{\bar{r}}) \geqslant 1/\bar{p}.$$

Obviously, we can find a positive integer $\bar{M}$ such that $\|\tilde{f}\|_{\mathscr{F}} \leqslant \bar{M}$. Therefore $\tilde{f} \in \Omega_{\bar{M}\bar{p}\bar{r}}$. Hence

$$\mathscr{A} \subset \bigcup_{M=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcup_{r=1}^{\infty} \Omega_{Mpr}.$$

Remark. Theorem 3 and a well-known Baire's theorem imply that the set $\mathscr{B}$ of those $\tilde{f} \in \mathscr{F}(P)$ for which successive approximations $\{z_n^f\}$ are convergent is dense and of second category in the space $(\mathscr{F}(P), \varrho)$.

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