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PRIME NUMBERS SUCH THAT THE SUMS OF THE DIVISORS OF THEIR POWERS ARE PERFECT SQUARES

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1. Introduction. For an integer $n \geqslant 1$ let $\sigma(n)$ be the sum of the divisors of n. For a fixed integer $\alpha \geqslant 1$ we consider the primes p for which

(1) $\sigma(p^\alpha)$ is a perfect square.

Using $\sigma(p^\alpha) = 1+p+ \ldots +p^\alpha$ one shows easily that for $\alpha = 1$ this can happen only for $p = 3$ and that for no prime p the sum $\sigma(p^2)$ is a square. Schinzel [3] showed that for $\alpha = 3$ we must have $p = 7$ and Thébault [5] considered $\alpha = 4$ showing that in this case $p = 3$.

On the other hand, by the celebrated general theorems of Siegel (see [2], Chapter 28, and [4]) and Baker [1] it is known that only finite number of primes satisfy (1) and, moreover, any such p satisfies

$$p < \exp\exp\exp(\alpha^{10\alpha^3}).$$

We prove

THEOREM. *Let α be an odd integer greater than or equal to 3. Then all prime numbers p such that $\sigma(p^\alpha)$ is a perfect square satisfy $p < 2^{2^{\alpha+1}}$.*

Numerical example. There is no prime p such that $\sigma(p^5)$ is a perfect square.

2. Lemmas. Let α be an integer greater than or equal to 3 and assume that $\sigma(p^\alpha) = m^2$. Then we have

$$p(1+p+ \ldots +p^{\alpha-1}) = (m-1)(m+1).$$

Hence $p \mid m+1$ or $p \mid m-1$.

LEMMA 1. *There exists an integer $x \neq 0$ such that*

$$2^6(1+p+ \ldots +p^{\alpha-3}) = p(px)^2+p\{-2(2^2-1)px+(2^3+1)\}+\varepsilon\{-2^3px+2^3(2^2-1)\}-2^4x,$$

where ε is $+1$ when $p \mid m+1$ and ε is -1 when $p \mid m-1$.

Proof. If $p \mid m+1$, then we write $m+1 = px_1$ $(x_1 \geqslant 1)$. We have

$$p(1+p+\ldots+p^{\alpha-2}) = px_1^2-(2x_1+1).$$

If we write $2x_1+1 = px_2$ $(x_2 \geqslant 1)$, then we have

$$2^2 p(1+p+\ldots+p^{\alpha-3}) = p^2 x_2^2-2px_2-(2^2 x_2+2^2-1).$$

If we write $2^2 x_2+2^2-1 = px_3$ $(x = x_3 \geqslant 1)$, then we obtain the formula of the lemma. If $p \mid m-1$, then we put $m-1 = px_1$, $2x_1-1 = px_2$, and $2^2 x_2+1-2^2 = -px$.

Remark. By Lemma 1, all primes p satisfying (1) are odd.

LEMMA 2. *If $3 \leqslant k \leqslant \alpha$, then the equation*

$$2^{\beta(k)}(1+p+\ldots+p^{\alpha-k}) = p^{k-2}(px_k)^2+\sum_{i=2}^{k} p^{k-i}\{\gamma_i^{(k)} px_k+\delta_i^{(k)}\}-2^{\mu(k)} x_k$$

holds, where $\beta(k)$, $\gamma_i^{(k)}$, $\delta_i^{(k)}$, $\mu(k)$ are integers that depend only on k and are independent of α, p, m. Moreover, $\beta(k)$, $\mu(k) > 0$, and x_k is an integer.

Proof. For $\alpha = 3$ this is Lemma 1. Assume that the lemma holds for $3 \leqslant k < \alpha$. Then, from

$$\begin{aligned} &2^{\beta(k)} p(1+p+\ldots+p^{\alpha-(k+1)}) \\ &= p^{k-2}(px_k)^2+\sum_{i=2}^{k-1} p^{k-i}\{\gamma_i^{(k)} px_k+\delta_i^{(k)}\}+\gamma_k^{(k)} px_k-(2^{\mu(k)} x_k+2^{\beta(k)}-\delta_k^{(k)}) \end{aligned}$$

we see that $p \mid 2^{\mu(k)} x_k+2^{\beta(k)}-\delta_k^{(k)}$. If we write

$$2^{\mu(k)} x_k+2^{\beta(k)}-\delta_k^{(k)} = px \qquad (x_{k+1} = x),$$

then

$$\begin{aligned} &2^{\beta(k)+2\mu(k)}(1+p+\ldots+p^{\alpha-(k+1)}) \\ &= p^{k-1}(px)^2+p^{k-1}\{-2(2^{\beta(k)}-\delta_k^{(k)})px+(2^{\beta(k)}-\delta_k^{(k)})^2\}+ \\ &\quad +p^{k-2}\{2^{\mu(k)}\gamma_2^{(k)} px-2^{\mu(k)}\gamma_2^{(k)}(2^{\beta(k)}-\delta_k^{(k)})\}+ \\ &\quad +p^{k-3}[2^{\mu(k)}\gamma_3^{(k)} px+\{-2^{\mu(k)}\gamma_3^{(k)}(2^{\beta(k)}-\delta_k^{(k)})+2^{2\mu(k)}\delta_2^{(k)}\}]+\ldots+ \\ &\quad +2^{\mu(k)}\gamma_k^{(k)} px+\{-2^{\mu(k)}\gamma_k^{(k)}(2^{\beta(k)}-\delta_k^{(k)})+2^{2\mu(k)}\delta_{k-1}^{(k)}\}-2^{2\mu(k)} x. \end{aligned}$$

Now we consider the properties of the constants $\beta(k)$, $\gamma_i^{(k)}$, $\delta_i^{(k)}$, and $\mu(k)$ constructed in the proof of Lemma 2. By the proof of Lemma 2 and Lemma 1, we have

LEMMA 3. *If $3 \leqslant k \leqslant \alpha$, then*

$$\mu(k) = 2^{k-1}, \qquad \beta(k) = 2^k-2,$$

$$\gamma_2^{(k+1)} = -2(2^{\beta(k)} - \delta_k^{(k)}),$$

$$\gamma_i^{(k+1)} = 2^{\mu(k)}\gamma_{i-1}^{(k)}, \quad i = 3, 4, \ldots, k+1,$$

$$\delta_2^{(k+1)} = (2^{\beta(k)} - \delta_k^{(k)})^2, \quad \delta_3^{(k+1)} = -2^{\mu(k)}\gamma_2^{(k)}(2^{\beta(k)} - \delta_k^{(k)}),$$

$$\delta_{i+1}^{(k+1)} = -2^{\mu(k)}\gamma_i^{(k)}(2^{\beta(k)} - \delta_k^{(k)}) + 2^{2\mu(k)}\delta_{i-1}^{(k)}, \quad i = 3, 4, \ldots, k.$$

LEMMA 4. *If* $3 \leqslant k \leqslant \alpha$ *and* $2 \leqslant i \leqslant k$, *then*

$$|\gamma_i^{(k)}| < 2^{1+2^k} \quad \text{and} \quad |\delta_i^{(k)}| < 2^{2^{k+1}-1}.$$

Proof. By the proof of Lemma 1 we have

$$\beta(3) = 6, \quad \mu(3) = 2^2, \quad \gamma_2^{(3)} = -2(2^2-1), \quad \gamma_3^{(3)} = \pm 2^3,$$

$$\delta_2^{(3)} = 1 + 2^3, \quad \delta_3^{(3)} = \pm 2^3(2^2-1).$$

Thus, for $k = 3$ the lemma holds. Let $k \geqslant 3$ and assume that the lemma holds for k. Then, by Lemma 3, we have

$$|\gamma_2^{(k+1)}| < 2(2^{2^k} + 2^{1+2+\ldots+2^k}) < 2^{1+2^{k+1}},$$

$$|\gamma_i^{(k+1)}| < 2^{2^{k-1}} \cdot 2^{1+2^k} < 2^{1+2^{k+1}} \quad (3 \leqslant i \leqslant k+1),$$

$$|\delta_2^{(k+1)}| < (2^{2^k} + 2^{1+2+\ldots+2^k})^2 < 2^{1+2+\ldots+2^{k+1}},$$

$$|\delta_i^{(k+1)}| < 2^{2^{k-1}} \cdot 2^{1+2^k}(2^{2^k} + 2^{1+2+\ldots+2^k}) + 2^{2^k} \cdot 2^{1+2+\ldots+2^k}$$

$$< 2^{1+2+\ldots+2^{k+1}} \quad (3 \leqslant i \leqslant k+1).$$

LEMMA 5. *If* α *is an odd integer greater than or equal to* 3 *and* $p > 2^{2^{\alpha+1}}$, *then* $x_k \neq 0$ *for* $k = 3, 4, \ldots, \alpha$.

Proof. Let k be the first r such that $x_r = 0$. By Lemma 1 we have $k > 3$. From Lemma 2 we obtain

$$2^{\beta(k)}(1 + p + \ldots + p^{\alpha-k}) = \sum_{i=2}^{k} \delta_i^{(k)} p^{k-i}.$$

Since α is odd, we have $\alpha - k \neq k-2$.

Case 1. $\alpha - k > k-2$. By Lemma 4, we have $|\delta_i^{(k)}| < p/2$. Hence $2^{\beta(k)} = \delta_k^{(k)} = \ldots = \delta_2^{(k)}$ and the coefficient $2^{\beta(k)}$ of $p^{\alpha-k}$ is zero, a contradiction.

Case 2. $\alpha - k < k-2$. Since $\delta_2^{(k)} = 0$, by Lemma 3 we have $2^{\beta(k-1)} = \delta_{k-1}^{(k-1)}$. Hence, by Lemma 2, we get

$$(2) \quad 2^{\beta(k-1)}(p + p^2 + \ldots + p^{\alpha-(k-1)})$$

$$= p^{k-3}(px_{k-1})^2 + \sum_{i=2}^{k-2} p^{k-1-i}\{\gamma_i^{(k-1)} px_{k-1} + \delta_i^{(k-1)}\} +$$

$$+ \gamma_{k-1}^{(k-1)} px_{k-1} - 2^{\mu(k-1)} x_{k-1}.$$

Consequently, $p \mid x_{k-1}$. If $x_{k-1} \neq 0$, then expressing both sides of (2) in the form $\sum_i a_i p^i$ ($|a_i| < p/2$) we see that the highest power of p on the right-hand side is $\alpha-(k-1) < k-1$ while the one on the left-hand side is equal to or greater than $k-3+2+2 = k+1$, a contradiction. Hence $x_{k-1} = 0$, which contradicts the choice of k.

3. Proof of the Theorem. Let $\alpha \geqslant 3$ and $p > 2^{2^{\alpha+1}}$. Assume $\sigma(p^\alpha) = m^2$. If we put $k = \alpha$ in the formula of Lemma 2, we have

$$p^\alpha x_\alpha^2 + \Big(\sum_{i=2}^{\alpha} p^{\alpha-i+1}\gamma_i^{(\alpha)} - 2^{\mu(\alpha)}\Big)x_\alpha + \Big(\sum_{i=2}^{\alpha} p^{\alpha-i}\delta_i^{(\alpha)} - 2^{\beta(\alpha)}\Big) = 0. \tag{3}$$

By Lemma 4, the discriminant D of the quadratic equation (3) satisfies

$$|D| \leqslant \Big(\sum_{i=2}^{\alpha} p^{\alpha-i+1}\cdot 2^{2^{\alpha+2}}\Big)^2 + 2^2 p^\alpha\Big(\sum_{i=2}^{\alpha} p^{\alpha-i}\cdot 2^{2^{\alpha+1}}\Big) < \alpha^2 p^{2(\alpha-1)}\cdot 2^{2^{\alpha+1+5}}.$$

Hence the root x_α of (3) satisfies $|x_\alpha| < \alpha \cdot 2^{2^{\alpha+4}} < 1$. By Lemmas 2 and 5, this is a contradiction.

4. Numerical example. If $\sigma(p^5) = m^2$, then $p \mid m+1$ or $p \mid m-1$.

Case 1. $p \mid m+1$. If we put $m+1 = px_1$, $2x_1+1 = px_2$, $4x_2+(2+1) = px_3$, $2x_3+2^2+1 = px_4$, and $2^3x_4+(2^5+2^2-1) = px$, then x is an integer not equal to 0 and

$$\begin{aligned} f_1(x) = {} & p^3(px)^2 + p^3[-2(1+2+2^5)px + (1+2^3+2^6+2^7+2^{10})] + \\ & + p^2[-2^4(2^2+1)px + 2^4(1+2+2^2+2^3+2^5+2^7)] + \\ & + p[-2^5(2^2-1)px + 2^5(1+2+2^3+2^4+2^7)] - \\ & - 2^7px - 2^8x - 2^7(2^6-1) = 0. \end{aligned}$$

The value x_0 such that $y = f_1(x_0)$ has the minimal value is given by

$$x_0 = (1+2+2^5)/p + 2^3(2^2+1)/p^2 + 2^4(2^2-1)/p^3 + 2^6/p^4 + 2^7/p^5 > 0.$$

If $p > 2^6 = 64$, then $x_0 < 2^6/p < 1$. Since $f_1(2^6/p) > 0$, we have $-1 < x < 1$. Therefore, in this case, if $p > 2^6$, then $\sigma(p^5)$ cannot be a perfect square.

Case 2. $p \mid m-1$. If we put $m-1 = px_1$, $2x_1-1 = px_2$, $2^2x_2+1-2^2 = -px_3$, $2x_3+2^3+2^2-1 = px_4$, and $2^3x_4+(2^6+2^5+2^2-1) = px$, then x is an integer not equal to 0 and

$$\begin{aligned} f_2(x) = {} & p^3(px)^2 + p^3[-2(2^6+2^5+2^2-1)px + (2^6+2^5+2^2-1)^2] + \\ & + p^2[-2^4(2^3+2^2-1)px + 2^4(2^{10}+2^6+1)] + \\ & + p[-2^5(2^2-1)px + 2^5(2^9+2^5-2^3+2+1)] + \\ & + 2^7px - 2^7(2^5+1) - 2^8x = 0. \end{aligned}$$

The value x_0 such that $y = f_2(x_0)$ has the minimal value is given by

$$x_0 = (2^6+2^5+2^2-1)/p+2^3(2^3+2^2-1)/p^2+2^4(2^2-1)/p^3-2^6/p^4+2^7/p^5 > 0.$$

If $p > 2^6+2^5+2^2$, then $x_0 < (2^6+2^5+2^2)/p < 1$. Since $f_2\left((2^6+2^5+2^2)/p\right) > 0$, we have $-1 < x < 1$. Therefore, in this case, the primes p such that $\sigma(p^5)$ is a perfect square satisfy $p < 2^6+2^5+2^2 = 100$.

From cases 1 and 2 we infer that the primes p such that $\sigma(p^5)$ is a perfect square satisfy $p < 100$. Using a computer, one checks that the values $1+p+p^2+p^3+p^4+p^5$ are not perfect squares for $3 \leqslant p \leqslant 97$.

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