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ON THE CONSTRUCTION OF NON-ISOMORPHIC STEINER QUADRUPLE SYSTEMS

BY

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1. Introduction. A *Steiner quadruple system* (or, more simply, a *quadruple system*) is a pair (Q, b), where Q is a finite set and b is a collection of 4-element subsets of Q (called *blocks*) such that any three distinct elements of Q belong to exactly one block of b. The number $|Q|$ is called the *order of the quadruple system* (Q, b). Hanani [4] proved in 1960 that the spectrum for quadruple systems consisted of the set of all positive integers $n \equiv 2$ or $4 \pmod 6$. It is easy to show that a quadruple system of order n has $n(n-1)(n-2)/24$ blocks. Two quadruple systems (Q_1, b_1) and (Q_2, b_2) are said to be *isomorphic* provided that there is a 1-1 mapping of Q_1 onto Q_2 which maps the blocks of b_1 onto the blocks of b_2. The purpose of this note is to give a very simple construction for quadruple systems which can be used to construct large numbers of non-isomorphic quadruple systems of a given order. Other work along these lines can be found in [1], [4], [8] and [10]. The techniques used in this construction are quite similar to those developed by the author in [5], [6] and [7].

2. Construction of Steiner quadruple systems. By a *3-skein* is meant a pair $(Q, \langle\, , ,\rangle)$, where Q is a finite set and $\langle\, , ,\rangle$ is a ternary operation on Q such that if in the equation $\langle x, y, z\rangle = w$ any three elements of x, y, z and w are given, then the remaining element is uniquely determined [2].

Now, let $(Q, b(q))$ and $(V, b(v))$ be quadruple systems based on $Q = \{1, 2, \ldots, q\}$ and $V = \{1, 2, \ldots, v\}$, respectively, and let $(Q, \langle\, , ,\rangle)$ be a 3-skein. Now define on the set $Q \times V$ the following collection b of 4-element subsets:

(1) For every block $\{a, b, c, d\} \in b(q)$ and for every $w \in V$, $\{(a, w), (b, w), (c, w), (d, w)\} \in b$.

(2) For every 2-element subset $\{a, b\}$ of Q and every 2-element subset $\{u, w\}$ of V, $\{(a, u), (b, u), (a, w), (b, w)\} \in b$.

(3) For every block $\{a, b, c, d\} \in b(q)$ and every 2-element subset $\{u, w\}$ of V the following six subsets belong to b:

$$\{(a, u), (b, u), (c, w), (d, w)\}, \quad \{(a, w), (b, u), (c, u), (d, w)\},$$
$$\{(a, u), (b, w), (c, u), (d, w)\}, \quad \{(a, w), (b, u), (c, w), (d, u)\},$$
$$\{(a, u), (b, w), (c, w), (d, u)\}, \quad \{(a, w), (b, w), (c, u), (d, u)\}.$$

(4) For every block $\{x, y, z, w\} \in b(v)$ and every three (not necessarily distinct) elements p, q and s of Q,

$$\{(p, x), (q, y), (s, z), (\langle p, q, s\rangle, w)\} \in b \quad \text{where } x < y < z < w.$$

THEOREM 1. *$(Q \times V, b)$ is a Steiner quadruple system.*

Proof. It suffices to show that every triple of distinct elements from $Q \times V$ belong to at least one block and that

$$|b| = (qv)(qv-1)(qv-2)/24.$$

So, let x, y and z be any three distinct elements of $Q \times V$. If x, y and z all have the same second coordinate, they belong to a block of type (1). If exactly two of them have the same second coordinate, they belong to a block of type (2) if the set of first coordinates contains two elements, and to a block of type (3) if all of the first coordinates are different. The last case is where x, y and z have distinct second coordinates. We may suppose that $x = (p, v)$, $y = (q, w)$ and $z = (s, u)$, where $v < w < u$. Let $\{v, w, u, t\}$ be the block of $b(v)$ containing v, w and u. One of 4 possibilities is true: $t < v < w < u$, $v < t < w < u$, $v < w < t < u$, $v < w < u < t$. We consider the case with $v < w < t < u$, the other cases being similar. Since the equation $\langle p, q, \bar{q}\rangle = s$ is uniquely solvable for $\bar{q}$ in Q,

$$\{(p, v), (q, w), (\bar{q}, t), (\langle p, q, \bar{q}\rangle = s, u)\} \in b$$

and, of course, x, y and z belong to this block. Hence, every triple of elements from $Q \times V$ belong to at least one block of b. On the other hand, by direct count, there are

$\left(\dfrac{q(q-1)(q-2)}{24}\right) v$ blocks of type (1),

$\dbinom{q}{2}\dbinom{v}{2}$ blocks of type (2),

$6\left(\dfrac{q(q-1)(q-2)}{24}\right)\dbinom{v}{2}$ blocks of type (3), and

$q^3\left(\dfrac{v(v-1)(v-2)}{24}\right)$ blocks of type (4).

Direct computation shows that the sum of these numbers is

$$[(vq)(vq-1)(vq-2)]/24 = |b|.$$

This completes the proof of the theorem.

We now modify the construction of $(Q\times V, b)$ by leaving blocks of types (1), (2) and (3) unchanged and by replacing blocks of type (4) as follows. Let

$$\{x, y, z, w\}_1, \{x, y, z, w\}_2, \ldots, \{x, y, z, w\}_t$$

be the $t = v(v-1)(v-2)/24$ blocks of $b(v)$ and let

$$(Q, \langle\,,\,,\rangle_1), (Q, \langle\,,\,,\rangle_2), \ldots, (Q, \langle\,,\,,\rangle_t)$$

be any $t = v(v-1)(v-2)/24$ 3-skeins. For every block $\{x, y, z, w\}_i$ and every three elements $p, q, s \in Q$, insert the block

$$\{(p, x), (q, y), (s, z), (\langle p, q, s\rangle_i, w)\}, \quad \text{where } x < y < z < w.$$

Denote this new system by $(Q\times V, b^*)$.

THEOREM 2. *$(Q\times V, b^*)$ is a Steiner quadruple system.*

3. Non-isomorphic Steiner quadruple systems. In this section we use the construction in Theorem 2 to produce large numbers of non-isomorphic quadruple systems of a given order. The following observations are crucial. A pair of 3-skeins $(Q, \langle\,,\,,\rangle_1)$ and $(Q, \langle\,,\,,\rangle_2)$ are *distinct* provided that $\langle p, q, s\rangle_1 \neq \langle p, q, s\rangle_2$ for at least one triple of elements p, q, s in Q. Since the operation table for a 3-skein is a latin cube, by starting with any latin square based on $Q = \{1, 2, \ldots, q\}$ and taking any cyclic permutation on Q, it is possible to construct a latin cube and, therefore, a 3-skein $(Q, \langle\,,\,,\rangle)$ having this latin cube as its operation table. Since there are at least $q!(q-1)!\ldots 2\cdot 1$ (the product of the first q factorials) distinct latin squares of order q [3], there are at least this many distinct latin cubes and, therefore, distinct 3-skeins of order q. We now are in a position to prove the following theorem:

THEOREM 3. *Let q and v be positive integers such that $q, v \equiv 2$ or 4 (mod 6). Then there are at least*

$$\frac{(q!(q-1)!\ldots 2\cdot 1)^t}{(qv)!}, \quad t = \frac{v(v-1)(v-2)}{24},$$

non-isomorphic Steiner quadruple systems of order qv.

Proof. A pair of quadruple systems (Q, b_1) and (Q, b_2) are *distinct* provided that there is at least one block of b_1 which does not belong to b_2. Now, let $(Q, b(q))$ and $(V, b(v))$ be quadruple systems of orders q and v, respectively. Let

$$(Q, \langle\,,\,,\rangle_1), (Q, \langle\,,\,,\rangle_2), \ldots, (Q, \langle\,,\,,\rangle_t)$$

be any $t = v(v-1)(v-2)/24$ 3-skeins. Since there are at least $q!(q-1)!\ldots 2\cdot 1$ distinct 3-skeins of order q, Theorem 2 gives at least $\big(q!(q-1)!\ldots 2\cdot 1\big)^t$ distinct quadruple systems of order qv. Since any isomorphism class can contain at most $(qv)!$ of these quadruple systems, the statement of the theorem follows.

4. Examples. Since $40 = 4\cdot 10$ and there are quadruple systems of order 4 and 10, Theorem 3 gives at least 10^{24} non-isomorphic quadruple systems of order 40. Since $64 = 4\cdot 16$, similar remarks produce at least 10^{255} non-isomorphic Steiner quadruple systems of order 64. Many more examples of this type are possible. These two examples give considerably stronger results than can be obtained by the results in [1].

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