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TWO SUFFICIENT CONDITIONS FOR THE MACLANE CLASS $\mathscr{A}$

BY

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Let $\mathscr{A}$ be the MacLane class of non-constant functions which are analytic in $|z| < 1$ and have asymptotic values in a dense set of points of $|z| = 1$. MacLane [3], p. 46, showed that if

$$M(r, f) = \sup_{|z|=r} |f(z)| \quad (0 < r < 1)$$

for a non-constant function $f(z)$ analytic in $|z| < 1$, then

$$\int_0^1 (1-r)\log^+ M(r, f)\, dr < \infty \tag{1}$$

is sufficient to guarantee $f(z) \in \mathscr{A}$. MacLane [3], p. 51, further proved that if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is such that, for some λ $(0 < \lambda < \frac{2}{3})$,

$$\log^+ |a_n| < n^\lambda \quad (n > n_0), \tag{2}$$

then $f(z) \in \mathscr{A}$. Thus, if order ϱ of $f(z)$, defined as

$$\varrho = \limsup_{r\to 1} \frac{\log^+ \log^+ M(r, f)}{-\log(1-r)},$$

satisfies $0 < \varrho < 2$, then $f(z) \in \mathscr{A}$.

Hornblower [2] weakened condition (1) and showed that if $f(z)$ is non-constant analytic in $|z| < 1$ such that

$$\int_0^1 \log^+ \log^+ M(r, f)\, dr < \infty, \tag{3}$$

then $f(z) \in \mathscr{A}$.

The purpose of the present paper is to weaken (2) and to obtain a sufficient condition on $|a_n/a_{n-1}|$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n} \qquad (a_n \neq 0 \text{ for all } n)$$

is in the class $\mathscr{A}$.

Our results imply that all non-constant functions, analytic in $|z| < 1$ and having finite order, are in the class $\mathscr{A}$. Further, we construct an example to show that there are functions of infinite order which also belong to the class $\mathscr{A}$.

LEMMA 1. *Let $f(z)$ be analytic and non-constant in $|z| < 1$ and let*

$$M(r,f) = \sup_{|z|=r} |f(z)|.$$

If, for some a $(1 < a < \infty)$,

$$\log^+\log^+ M(r,f) = O\left\{(1-r)^{-1}\left(\log\frac{e}{1-r}\right)^{-a}\right\} \quad \text{as } r \to 1,$$

then $f(z) \in \mathscr{A}$.

Proof. It is easily seen that the hypothesis of the lemma implies

$$\int_0^1 \log^+\log^+ M(r,f)\,dr < \infty.$$

$f(z) \in \mathscr{A}$ now follows from Hornblower's result ([2], Theorem 1).

THEOREM 1. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n} \qquad (|z| < 1)$$

be a non-constant function. If, for some β $(1 < \beta < \infty)$,

$$\log^+|a_n| = O\{\lambda_n(\log\lambda_n)^{-1}(\log\log\lambda_n)^{-\beta}\} \quad \text{as } n \to \infty, \tag{4}$$

then $f(z) \in \mathscr{A}$.

Proof. Let us first observe that condition (4) implies that $f(z)$ is analytic in $|z| < 1$ and that there exist positive finite constants B and S such that, for all $n > S$,

$$\log^+|a_n| < B\lambda_n(\log\lambda_n)^{-1}(\log\log\lambda_n)^{-\beta}.$$

We write

$$M(r,f) \leqslant \sum_{n=0}^{\infty}|a_n| r^{\lambda_n} = \sum_{n=0}^{S}|a_n| r^{\lambda_n} + \sum_{n=S+1}^{N}|a_n| r^{\lambda_n} + \sum_{n=N+1}^{\infty}|a_n| r^{\lambda_n}, \tag{5}$$

where

$$N = \left[\exp\left\{\exp\left(\frac{1}{2B}\log\frac{1}{r}\right)^{-1/(\beta+1)}\right\}\right].$$

It follows that

$$\sum_{n=N+1}^{\infty} |a_n| r^{\lambda_n} = o(1) \quad \text{as } r \to 1,$$

for

$$\sum_{n=N+1}^{\infty} |a_n| r^{\lambda_n} < \sum_{n=N+1}^{\infty} \exp\{B\lambda_n(\log\lambda_n)^{-1}(\log\log\lambda_n)^{-\beta}\} r^{\lambda_n}$$

$$\leqslant \sum_{n=N+1}^{\infty} r^{\lambda_n/2} \leqslant \frac{r^{(N+1)/2}}{1-r^{1/2}},$$

and $r^{(N+1)/2}/(1-r^{1/2}) \to 0$ as $r \to 1$ in view of the estimate

$$1-r = \left(\log\frac{1}{r}\right)\left[1+O\left(\log\frac{1}{r}\right)\right]$$

for the values of r sufficiently close to 1. Thus, by (5), for all r satisfying $r_0 < r < 1$,

(6) $$M(r,f) < c(S) + N \max_{n\geqslant 0}\left\{\exp\left(B\lambda_n(\log\lambda_n)^{-1}(\log\log\lambda_n)^{-\beta}\right) r^{\lambda_n}\right\} + o(1),$$

where $c(S)$ is a constant depending on S. Now, let

(7) $$g(x, r) = Bx(\log x)^{-1}(\log\log x)^{-\beta} + x\log r.$$

The maximum value of $g(x, r)$ occurs at the point $x = x_0 \equiv x_0(r)$ satisfying the equation

$$B(\log x)^{-1}(\log\log x)^{-\beta}\{1-(\log x)^{-1} - B\beta(\log x)^{-1}(\log\log x)^{-1}\} = \log\frac{1}{r}.$$

It is easily seen that $x_0(r) \to \infty$ as $r \to 1$, so that, for all r satisfying $r_1 < r < 1$ we have

$$x_0(r) = \exp\left\{(1+o(1))B\left(\log\frac{1}{r}\right)^{-1}\left(-\log\log\frac{1}{r}\right)^{-\beta}\right\}.$$

Thus, by (7),

$$g(x, r) \leqslant Bx_0(\log x_0)^{-1}(\log\log x_0)^{-\beta}\{1+B\beta(\log\log x_0)^{-1}\}.$$

Using the estimate of $x_0(r)$, this estimate of $g(x, r)$ yields

$$\log g(x, r) \leqslant \log x_0 + o(1)$$
$$= B\big(1+o(1)\big)\left(\log\frac{1}{r}\right)^{-1}\left(-\log\log\frac{1}{r}\right)^{-\beta} + o(1)$$

for all values of r sufficiently close to 1. Now it follows from (6) that, as $r \to 1$,

$$\log^+ M(r, f) < \log N + \exp\left\{B\big(1+o(1)\big)\left(\log\frac{1}{r}\right)^{-1}\left(-\log\log\frac{1}{r}\right)^{-\beta} + o(1)\right\} +$$
$$+ o(1) \leqslant \exp\left(\frac{1}{2B}\log\frac{1}{r}\right)^{-1/(\beta+1)} +$$
$$+ \exp\left\{B\big(1+o(1)\big)\left(\log\frac{1}{r}\right)^{-1}\left(-\log\log\frac{1}{r}\right)^{-\beta} + o(1)\right\} + o(1)$$
$$= \big(1+o(1)\big)\exp\left\{B\big(1+o(1)\big)\left(\log\frac{1}{r}\right)^{-1}\left(-\log\log\frac{1}{r}\right)^{-\beta} + o(1)\right\}.$$

Since the right-hand side expression in this inequality is a positive quantity, we have, as $r \to 1$,

$$\log^+\log^+ M(r, f) \leqslant B\big(1+o(1)\big)\left(\log\frac{1}{r}\right)^{-1}\left(-\log\log\frac{1}{r}\right)^{-\beta} + o(1)$$
$$= O\left\{\left(\log\frac{1}{r}\right)^{-1}\left(-\log\log\frac{1}{r}\right)^{-\beta}\right\} = O\left\{(1-r)^{-1}\left(\log\frac{e}{1-r}\right)^{-\beta}\right\}.$$

Thus, by Lemma 1, $f(z) \in \mathscr{A}$ and the proof of Theorem 1 is complete.

Remark. Since condition (4) follows from MacLane's condition (2), Theorem 1 provides an improvement to MacLane's result (see [3], p. 51). Further, if

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$$

is analytic in $|z| < 1$ and has order ϱ, then proceeding on the lines of Beuermann [1] or MacLane [3] (p. 47) it is not difficult to prove that

$$\frac{\varrho}{1+\varrho} = \limsup_{n\to\infty} \frac{\log^+\log^+|a_n|}{\log\lambda_n}. \tag{8}$$

It is clear from (8) that, for functions of finite order, condition (4) is satisfied for any $\beta > 1$. Thus, by Theorem 1, all non-constant functions, analytic in $|z| < 1$ and having finite order, are in $\mathscr{A}$. The same conclusion also follows by Hornblower's result, since a function analytic in $|z| < 1$ and having finite order satisfies (3).

An example of a function in the class $\mathscr{A}$ having infinite order can easily be constructed by help of Theorem 1. Indeed, consider the function

$$g(z) = \sum_{n=0}^{\infty} \exp\left(\lambda_n (\log \lambda_n)^{-3}\right) z^{\lambda_n},$$

where $\lambda_0 = 0$ and $\{\lambda_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers. Since (4) is satisfied, we infer that $g(z)$ is in the class $\mathscr{A}$, and, by (8), the order of $g(z)$ is infinite.

Let us observe that whereas Theorem 1 provides an example of a function of infinite order in the class $\mathscr{A}$ in terms of a Gap Taylor series, such examples in the closed form can easily be constructed with the use of Hornblower's result. Indeed, the function

$$h(z) = \exp\left(\exp(1-z)^{-a}\right) \quad (0 < a < 1)$$

is in the class $\mathscr{A}$ in view of (3), and it can easily be seen that the order of $h(z)$ is infinite.

LEMMA 2. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$$

be a non-constant function analytic in $|z| < 1$ *and of order* ϱ. *If*

$$\psi(n) = \left|\frac{a_n}{a_{n+1}}\right|^{1/(\lambda_{n+1}-\lambda_n)} \geqslant \frac{1}{e} \quad \textit{for all } n > n_0,$$

then

$$1+\varrho \leqslant \max(1, \theta), \tag{9}$$

where

$$\theta = \limsup_{n\to\infty} \frac{\log \lambda_n}{\log\left((\lambda_n - \lambda_{n-1})/\log^+ |a_n/a_{n-1}|\right)}.$$

Proof. The condition $\psi(n) \geqslant 1/e$ for all $n > n_0$ implies $0 \leqslant \theta \leqslant \infty$.

Let $\theta < \infty$. For any δ such that $\theta < \delta < \infty$ we have, for all $n \geqslant N = N(\delta)$,

$$\log^+ \left|\frac{a_n}{a_{n-1}}\right| < (\lambda_n - \lambda_{n-1}) \lambda_n^{-1/\delta}.$$

Therefore, if $n > \max(N, n_0)$, then

$$\begin{aligned} \log|a_n| &< \log|a_N| + (\lambda_{N+1} - \lambda_N)\lambda_{N+1}^{-1/\delta} + \ldots + (\lambda_n - \lambda_{n-1})\lambda_n^{-1/\delta} \\ &= \log|a_N| + \lambda_n^{(\delta-1)/\delta} - \sum_{m=N+1}^{n-1} \lambda_m (\lambda_{m-1}^{-1/\delta} - \lambda_m^{-1/\delta}) - \lambda_N \lambda_{N+1}^{-1/\delta} \\ &= \log|a_N| + \lambda_n^{(\delta-1)/\delta} - \int_{\lambda_{N+1}}^{\lambda_n} n(t)\, d(t^{-1/\delta}) - \lambda_N \lambda_{N+1}^{-1/\delta}, \end{aligned} \tag{10}$$

where $n(t) = \lambda_m$ for $\lambda_m < t \leqslant \lambda_{m+1}$ and $m = N+1, \ldots, n-1$.

Since

$$\int_{\lambda_{N+1}}^{\lambda_n} n(t)\,d(t^{-1/\delta}) > -\frac{1}{\delta}\int_{\lambda_{N+1}}^{\lambda_n} t^{-1/\delta}dt = -\frac{1}{\delta-1}\{\lambda_n^{(\delta-1)/\delta}-\lambda_{N+1}^{(\delta-1)/\delta}\},$$

equation (10), for sufficiently large values of n, gives

$$\log|a_n| < \log|a_N| + \frac{\delta}{\delta-1}\,\lambda_n^{(\delta-1)/\delta} - \frac{1}{\delta-1}\,\lambda_{N+1}^{(\delta-1)/\delta} - \lambda_N\,\lambda_{N+1}^{-1/\delta}. \tag{11}$$

If $\theta < \delta < 1$, then (11) and (8) imply $\varrho = 0$, and so (9) obviously holds. Hence, suppose that $1 \leqslant \theta < \delta < \infty$. It follows from (11) that, for sufficiently large values of n,

$$\log^+\log^+|a_n| < \frac{\delta-1}{\delta}\log\lambda_n + o(1),$$

which, in view of (8), implies the inequality

$$\frac{\varrho}{1+\varrho} \leqslant \frac{\delta-1}{\delta}.$$

Since this inequality holds for every $\delta > \theta$, we have

$$\frac{\varrho}{1+\varrho} \leqslant \frac{\theta-1}{\theta},$$

and so $1+\varrho \leqslant \theta$. This completes the proof.

THEOREM 2. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$$

be a non-constant function. If, for some η $(0 < \eta < \infty)$,

$$\log^+\left|\frac{a_n}{a_{n-1}}\right| = O\big((\lambda_n-\lambda_{n-1})\lambda_n^{-\eta}\big) \quad \text{as } n \to \infty, \tag{12}$$

then $f(z) \in \mathscr{A}$.

Proof. It is easily seen that $f(z)$ is analytic in $|z| < 1$. In view of Lemma 2, condition (12) on $|a_n/a_{n-1}|$ implies that $f(z)$ is of finite order. The assertion $f(z) \in \mathscr{A}$ now follows from the remark following Theorem 1.

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