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SYMMETRIC OPERATIONS IN GROUPS

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Introduction. We say that an operation f on A (i.e., a function $f\colon A^n \to A$) is *generated* by a set $\boldsymbol{F}$ of operations on A, if f is a composition of some operations belonging to $\boldsymbol{F}$ and some trivial operations (= identity operations).

Let G be a group. We denote by $A^{(n)}(G)$ the set of all operations on the set G which are generated by the operations xy and x^{-1}, or, in other words, the set of all n-ary algebraic operations in G (see [1]), or else, the set of all words of n variables $x_1, \ldots, x_n$. The set $A^{(n)}(G)$ forms a group, the multiplication being defined by juxtaposition. In this group we distinguish the subgroup of all symmetric operations $S^{(n)}(G)$, that is the set of all words $s(x_1, \ldots, x_n)$ for which the equation

$$s(x_1, x_2, \ldots, x_n) = s(x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma n})$$

holds for every $x_1, x_2, \ldots, x_n \in G$ and for all permutations $\sigma \in S_n$.

The purpose of this paper is to study symmetric operations and the possibilitity of generating the group operation xy by symmetric operations of many (in general) variables. The class of groups in which this turns out to be possible we denote by $\mathscr{K}$.

In section I we give a complete description of $S^{(n)}(G)$ for nilpotent groups of class 2 and for arbitrary n, and, in section II, for normal products of Z_p and Z_2 for $n = 2$.

In section III we investigate the class $\mathscr{K}$. It is clear that abelian groups belong to $\mathscr{K}$, and E. Marczewski (cf. [2]) raised a question whether these are the only groups in $\mathscr{K}$. Unexpectedly enough, it turns out (see section IV) that $\mathscr{K}$ contains the symmetric group on three letters S_3. This leaves an open question of giving a more accurate description of the class $\mathscr{K}$ (**P 684**).

I. Nilpotent groups of class 2. Now we are going to determine the symmetric operations in the nilpotent groups of class 2. Let us recall the well-known identity

$$[x^n, y] = [x, y^n] = [x, y]^n. \tag{1}$$

THEOREM 1. *If G is a nilpotent group of class 2, then operation $f \in A^{(n)}(G)$ is symmetric if and only if*

$$f(x_1, x_2, \ldots, x_n) = x_1^a x_2^a \ldots x_n^a \prod_{1 \leqslant j < i \leqslant n} [x_i, x_j]^b, \tag{2}$$

where a, b are integers and

$$a^2 \equiv 2b (\exp G'). \tag{3}$$

Proof. Every word f in G is of the form

$$f(x_1, x_2, \ldots, x_n) = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \prod_{1 \leqslant j < i \leqslant n} [x_i, x_j]^{b_{ij}}$$

and the condition $f(x_1, x_2, \ldots, x_n) = f(x_2, x_1, x_3, \ldots, x_n)$ together with (1) yields

$$x_1^{a_1} \ldots x_n^{a_n} \prod_{1 \leqslant j < i \leqslant n} [x_i, x_j]^{b_{ij}}$$

$$= x_2^{a_1} x_1^{a_2} x_3^{a_3} \ldots x_n^{a_n} \prod_{3 \leqslant j < i \leqslant n} [x_i, x_j]^{b_{ij}} \prod_{3 \leqslant i \leqslant n} [x_i, x_1]^{b_{i2}} \prod_{3 \leqslant i \leqslant n} [x_i, x_2]^{b_{i1}} [x_1, x_2]^{b_{21}}$$

$$= x_1^{a_2} x_2^{a_1} x_3^{a_3} \ldots x_n^{a_n} \prod_{3 \leqslant i \leqslant n} [x_i, x_1]^{b_{i2}} \prod_{3 \leqslant i \leqslant n} [x_i, x_1]^{b_{i1}} \prod_{3 \leqslant j < i \leqslant n} [x_i, x_j]^{b_{ij}} [x_2, x_1]^{a_1 a_2 - b_{21}}.$$

Hence we have

$$a_1 \equiv a_2 (\exp G'), \tag{4}$$

$$a_1 a_2 \equiv 2b_{21} (\exp G'), \tag{5}$$

$$b_{i1} \equiv b_{i2} (\exp G'), \quad i = 3, 4, \ldots, n. \tag{6}$$

From the condition $f(x_1, \ldots, x_n) = f(x_2, x_3, \ldots, x_n, x_1)$ we infer by (1) that

$$x_1^{a_1} \ldots x_n^{a_n} \prod_{1 \leqslant j < i \leqslant n} [x_i, x_j]^{b_{ij}}$$

$$= x_2^{a_1} x_3^{a_2} \ldots x_n^{a_{n-1}} x_1^{a_n} \prod_{1 \leqslant j < i < n} [x_{i+1}, x_{j+1}]^{b_{ij}} \prod_{1 \leqslant j < n} [x_1, x_{j+1}]^{b_{nj}}$$

$$= x_1^{a_n} x_2^{a_1} \ldots x_n^{a_{n-1}} \prod_{2 \leqslant i \leqslant n} [x_i, x_1]^{a_{n-1} a_n} \prod_{1 \leqslant j < i < n} [x_{i+1}, x_{j+1}]^{b_{ij}} \prod_{1 \leqslant j < n} [x_{j+1}, x_1]^{-b_{nj}}.$$

This yields

$$a_1 \equiv a_2 \equiv \ldots \equiv a_n (\exp G), \tag{7}$$

$$b_{ij} \equiv b_{i+1, j+1} (\exp G') \quad \text{for } 1 \leqslant j < i < n, \tag{8}$$

$$b_{i1} + b_{n, i-1} \equiv a_{i-1} a_n (\exp G') \quad \text{for } 2 \leqslant i \leqslant n. \tag{9}$$

Now we shall prove the theorem by induction on n. For $n = 1, 2, 3$ our statement readily follows from (4), (5) and (6). Suppose that (4)-(9) imply (2) and (3) for $n-1$ ($n \geqslant 4$). This means that

$$a_i \equiv a(\exp G) \quad \text{for } 1 \leqslant i \leqslant n-1,$$
$$b_{ij} \equiv b(\exp G') \quad \text{for } 1 \leqslant j < i \leqslant n-1,$$
$$a^2 \equiv 2b(\exp G').$$

In view of (7) we have $a_n \equiv a(\exp G)$, while for k such that $1 < k < n$ the relation $b_{ni} \equiv b_{n-1,i-1}(\exp G')$ follows from (8). Now using (6) for $i = n$ as well as the induction hypothesis, we conclude that every n-ary symmetric operation must be of the form (2), and (3) holds.

If $a_i \equiv a(\exp G)$, $b_{ij} \equiv b(\exp G')$ for $1 \leqslant i < j \leqslant n$ and (3) is satisfied, then (4)-(9) are satisfied too. And since the cycles $(1, 2)$ and $(1, 2, \ldots, n)$ generate the symmetric group S_n, f is symmetric. Thus the proof is completed.

II. Normal products Z_pZ_2. Let us consider the normal product Z_pZ_2 of a cyclic group Z_p (for a prime $p > 2$) and the group Z_2, i.e. the group of pairs (ε, k), where $\varepsilon = +1$ or -1, $k \in Z_p$, and the multiplication being defined by the equality

$$(\varepsilon, k)(\eta, l) = (\varepsilon\eta, \eta k + l).$$

For the sake of brevity we write k instead of $(1, k)$ and kb instead of $(-1, k)$, $k \in Z_p$.

Let us begin with the two simple facts:

(i) *The commutator subgroup of Z_pZ_2 is Z_p.*

(ii) *For every $0 < i < p$ and $0 \leqslant j < p$ there exists precisely one automorphism φ of Z_pZ_2 for which $\varphi(1) = i$ and $\varphi(0b) = jb$.*

To verify (ii) define

$$\varphi(kb) = (j + ki)b, \quad \varphi(k) = ki \quad \text{for } 0 \leqslant k < p, \tag{10}$$

and check that φ is an automorphism of Z_pZ_2.

Now we prove the following useful

LEMMA. *Let $w, w' \in A^{(2)}(Z_pZ_2)$. If w, w' are equal on the pairs $\langle 1, 0\rangle$, $\langle 0, 1\rangle$, $\langle 1, 0b\rangle$, $\langle 0b, 1\rangle$, $\langle 0b, 1b\rangle$, $\langle 1b, 0b\rangle$, then w, w' are identical everywhere in Z_pZ_2.*

Proof. Let

$$w = x^{\alpha_1}y^{\beta_1} \ldots x^{\alpha_n}y^{\beta_n}, \qquad w' = x^{\gamma_1}y^{\delta_1} \ldots x^{\gamma_m}y^{\delta_m}.$$

If $w(1, 0) = w'(1, 0)$ and $w(0, 1) = w'(0, 1)$, then

$$\sum \alpha_i \equiv \sum \gamma_i(p), \qquad \sum \beta_i \equiv \sum \delta_i(p),$$

and therefore for every k, l with $0 \leqslant k,\ l < p$ we have

$$w(k,l) = k^{\Sigma\alpha_i} l^{\Sigma\beta_i} \equiv k^{\Sigma\gamma_i} l^{\Sigma\delta_i} = w'(k,l)(p).$$

Since w and w' must commute with each automorphism φ of Z_pZ_2, therefore if

$$\varphi(1) = k, \quad \varphi(0b) = k'b,$$

then we have

$$w(k, k'b) = w'(k, k'b) \quad \text{and} \quad w(k'b, k) = w'(k'b, k)$$

for all k, k' such that $0 < k < p$, $0 \leqslant k' < p$.

If $0 \leqslant k' < k < p$, then the mapping φ defined by

$$\varphi(1) = k'-k, \quad \varphi(0b) = kb$$

is, by (ii), an automorphism of Z_pZ_2, and

$$\varphi(1b) = (k+k'-k)b = k'b.$$

Hence

$$w(kb, k'b) = w'(kb, k'b) \quad \text{and} \quad w(k'b, kb) = w'(k'b, kb)$$

for any k, k' with $0 \leqslant k < k' < p$.

Further, the mapping

(11) $$e\colon Z_pZ_2 \to Z_pZ_2,$$

where $e(k) = 0$, and $e(kb) = 0b$ for $0 \leqslant k < p$, is an endomorphism of Z_pZ_2, and thus

$$ew(1, 0b) = w(0, 0b) = w'(0, 0b) = ew'(1, 0b),$$

$$ew(0b, 1) = w(0b, 0) = w'(0b, 0) = ew'(0b, 1).$$

Hence

$$w(0, kb) = w'(0, kb) \quad \text{and} \quad w(kb, 0) = w'(kb, 0)$$

because kb $(1 \leqslant k < p)$ is an image of $0b$ by an automorphism. Finally, we have

$$w(kb, kb) = w(kb, 0)w(0, kb) = w'(kb, 0)w'(0, kb) = w'(kb, kb)$$

for all k $(0 \leqslant k < p)$.

The following theorem gives a description of the symmetric binary words in Z_pZ_2.

THEOREM 2. *We have*

$$S^{(2)}(Z_pZ_2) = \mathrm{gp}\{w_p, u\} \cong Z_p \times Z_pZ_2,$$

where

(12) $$w_p(x, y) = xy[y, x]^{(p+1)/2}, \quad u(x, y) = x^2y^2.$$

Proof. Since

$$w_p(y, x) = yx[x, y]^{(p+1)/2} = xy[y, x]^{-(p+1/2)+1} = xy[y, x]^{(p+1)/2} = w_p(x, y),$$

$$u(y, x) = y^2x^2 = x^2y^2 = u(x, y),$$

we get the inclusion

$$\mathrm{gp}\{w_p, u\} \subset \boldsymbol{S}^{(2)}(Z_pZ_2).$$

Now we show that if $s \in \boldsymbol{S}^{(2)}(Z_pZ_2)$, then

$$s(0b, 1b) = 0. \tag{13}$$

If $s(x, y) = x^{\alpha_1}y^{\beta_1} \dots x^{\alpha_n}y^{\beta_n}$, then from $s(0, 0b) = s(0b, 0)$ we obtain

$$\sum \alpha_i \equiv \sum \beta_i (2).$$

Consequently,

$$s(0b, 0b) = 0b^{\Sigma\alpha_i}0b^{\Sigma\beta_i} = 0b^{\Sigma\alpha_i+\Sigma\beta_i} = 0. \tag{14}$$

Because s commutes with the endomorphism e defined in (11), the equality (14) implies

$$0 = s(0b, 0b) = s\big(e(0b), e(1b)\big) = es(0b, 1b),$$

and thus $s(0b, 1b) \in Z_p$.

Let us suppose that $s(0b, 1b) = k$, and consider an automorphism $\varphi(1) = -1, \varphi(0b) = 1b$. Hence we get

$$\varphi(1b) = \varphi(0b\cdot 1) = \varphi(0b)\cdot\varphi(1) = 1b\cdot(-1) = 0b$$

and, furthermore,

$$k = s(0b, 1b) = s(1b, 0b) = s\big(\varphi(0b), \varphi(1b)\big) = \varphi(k).$$

One can see that the only $k \in Z_p$ for which the equality $\varphi(k) = k$ holds is equal to 0. Therefore $s(0b, 1b) = 0$.

Let us consider the mapping a: $\boldsymbol{S}^{(2)}(Z_pZ_2) \to Z_p \times Z_pZ_2$ defined by

$$a(s) = \langle s(1, 0), s(1, 0b)\rangle.$$

Since

$$\begin{aligned} a(s_1s_2) &= \langle s_1s_2(1, 0), s_1s_2(1, 0b)\rangle \\ &= \langle s_1(1, 0), s_1(1, 0b)\rangle\langle s_2(1, 0), s_2(1, 0b)\rangle = a(s_1)a(s_2), \end{aligned}$$

a is a homomorphism. Moreover, since $s(0b, 1b) = 0$ for all $s \in \boldsymbol{S}^{(2)}$, therefore, if $s_1 \neq s_2$, then, by the lemma, either $s_1(1, 0) \neq s_2(1, 0)$ or $s_1(1, 0b) \neq s_2(1, 0b)$. This means that the mapping a is one-to-one. Observe now that

$$a(w_p) = \langle 1, 0b\rangle \quad \text{and} \quad a(u) = \langle 2, 2\rangle$$

are the generators of $Z_p \times Z_p Z_2$, and therefore

$$\mathrm{gp}\{w_p, u\} = S^{(2)}(Z_p Z_2) \cong Z_p \times Z_p Z_2.$$

This completes the proof.

III. The class $\mathscr{K}$.

THEOREM 3. *The class $\mathscr{K}$ is closed under taking subgroups, homomorphism images, and direct powers.*

Proof. Observe that if s is a symmetric operation in G, then s is symmetric in any group of the variety of groups, i.e. in the $HSP(G)$ generated by G. If $G \in \mathscr{K}$, then the operation xy is generated by symmetric operations, and the equation expressing this fact is satisfied in any group of $HSP(G)$.

THEOREM 4. *If a nilpotent group G belongs to $\mathscr{K}$, then G is abelian.*

Proof. In view of Theorem 3 it is sufficient to prove Theorem 4 for nilpotent group of class 2. To do this we show(1) that if $s \in S^{(n)}(G)$, then

$$s(x_1^{-1}, \ldots, x_n^{-1}) = s(x_1, \ldots, x_n)^{-1}. \tag{15}$$

By theorem 1,

$$s(x_1, \ldots, x_n) = x_1^a \ldots x_n^a \prod_{1 \leqslant j < i \leqslant n} [x_i, x_j]^b, \qquad \text{where } a^2 \equiv 2b(\exp G'), \tag{16}$$

whence

$$\begin{aligned} s^{-1}(x_1, \ldots, x_n) &= \prod_{1 \leqslant j < i \leqslant n} [x_i, x_j]^{-b} \cdot x_n^{-a} \ldots x_1^{-a} \\ &= x_1^{-a} \ldots x_n^{-a} \prod_{1 \leqslant j < i \leqslant n} [x_i^{-a}, x_j^{-a}] \prod_{1 \leqslant j < i \leqslant n} [x_i, x_j]^{-b}. \end{aligned}$$

Hence, by (16), we get (15).

If algebraic operations $s_1, \ldots, s_k$ satisfy (15), then so does the operation $s_1(s_2, \ldots, s_k)$. Hence, since $G \in \mathscr{K}$, we have $x^{-1}y^{-1} = (xy)^{-1}$, which implies that G is abelian.

Theorems 3 and 4 produce an aboundance of groups which are not in $\mathscr{K}$. For example, we have the following

COROLLARY. *If a finite group G has a non-abelian Šylov subgroup, then G is not in $\mathscr{K}$. Consequently, $S_n \notin \mathscr{K}$ for $n \geqslant 4$.*

IV. A non-abelian group in $\mathscr{K}$.

In this section we show that $S_3 \in \mathscr{K}$.

THEOREM 5. *In $Z_3 Z_2$ we have*

$$xy = w_3\big[w_3 u\big(w_3 u(x, y), y^4\big), w_3\big(w_3^4(x, y), s(x, y, x)\big)\big], \tag{17}$$

(1) The idea of this proof is due to S. Fajtlowicz.

where $w_3(x,y) = xy[x,y]$, $u(x,y) = x^2y^2$, *and* $s(x,y,z) = [z,y,x]\times\times[x,y,z]$ *is a ternary symmetric operation in* Z_3Z_2. *Consequently,* $Z_3Z_2 \in \mathcal{K}$.

Proof. First we check that s is symmetric. In fact, by virtue of Jacobi identity valid in meta-abelian groups, we have

$$\begin{aligned} s(y,x,z) &= [z,x,y][y,x,z] = [x,y,z]^2[x,y,z]^2[y,z,x]^2 \\ &= [z,y,x][x,y,z] = s(x,y,z), \\ s(y,z,x) &= [x,z,y][y,z,x] = [y,z,x][y,x,z]^2[z,y,x]^2 \\ &= [z,y,x][x,y,z] = s(x,y,z). \end{aligned}$$

To prove (17) we apply lemma and verify that:

$$\begin{aligned} R(0,1) &= w_3\big(w_3u(0,1), w_3(1,0)\big) = w_3(0,1) = L(0,1), \\ R(1,0) &= w_3\big(w_3u(0,0), w_3(1,0)\big) = w_3(0,1) = L(1,0), \\ R(1,0b) &= w_3\big(w_3u(2b,0), w_3(0,0)\big) = w_3(2b,0) = 2b = L(1,0b), \\ R(0b,1) &= w_3\big(w_3u(2b,1), w_3(0,1)\big) = w_3(1b,1) = 1b = L(0b,1), \\ R(0b,1b) &= w_3\big(w_3u(0,0), w_3(0,1)\big) = w_3(0,1) = L(0b,1b), \\ R(1b,0b) &= w_3\big(w_3u(0,0), w_3(0,2)\big) = w_3(0,2) = L(1b,0b). \end{aligned}$$

This completes the proof.

It is of interest that *in* S_3 *the operation* xy *is not generated by the set of binary symmetric operations.* More precisely, we prove

THEOREM 6. *Every algebraic binary operation* f *from the algebra*

$$\mathfrak{A} = \langle Z_pZ_2; \boldsymbol{S}^{(1)} \cup \boldsymbol{S}^{(2)}\rangle$$

satisfies

$$(18) \qquad f(ib, jb) \in \{0, 0b, 1b, \ldots, (p-1)b\} = B, \quad 0 \leqslant i,j < p.$$

Consequently, the operation xy *does not belong to* $\boldsymbol{A}^{(2)}(\mathfrak{A})$.

Proof. Since every element of B is of order $\leqslant 2$, every unary operation maps B into B. If $f \in \boldsymbol{S}^{(2)}(Z_pZ_2)$, then, by Theorem 2, the operation f is of the form $w_p^k u^l$ with $0 \leqslant k < 2p$, $0 \leqslant l < p$. We have

$$\begin{aligned} w_p(ib,jb) &= ib\cdot jb\cdot[jb,ib]^{(p+1)/2} = (j-i)[(i-j)(i-j)]^{(p+1)/2} = 0, \\ u(ib,jb) &= 0. \end{aligned}$$

Suppose now that f_1 and f_2 satisfy (18) and consider the superpositions $w_p(f_1,f_2)$ and $u(f_1,f_2)$. We see that

$$w_p\big(f_1(ib,jb), f_2(ib,jb)\big) \in B,$$

$$u\big(f_1(ib,jb), f_2(ib,jb)\big) = 0,$$

whence

$$w_p^k u^l\big(f_1(ib,jb), f_2(ib,jb)\big) \in B,$$

and (18) follows.

Since $ib \cdot jb = j - i$, the operation xy cannot be an algebraic operation in $\mathfrak{A}$.

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The results of this paper were announced in [3].

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