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BOREL SETS OF EXACT CLASS

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The existence of Borel sets of exact class in various metric spaces is classical and well known. In [2], the authors give an elegant construction of sets M_a ($a < \omega_1$) of exact multiplicative class a in the Hilbert cube. Their construction can also be extended to the Cantor set C, from which it follows that every metric space containing C contains Borel sets of exact class for each $a < \omega_1$. Their construction appeals to the Brouwer fixed-point theorem for the Hilbert cube. The purpose of this note is to show how a similar construction can be effected appealing only to the elementary contraction mapping theorem. For convenience, we carry out our construction in the Cantor set C rather than the Hilbert cube.

Recall that the family $\mathfrak{B}$ of Borel sets of a metrizable space X can be constructed by transfinite induction. Let

$$\mathfrak{G}_0 = \{G\colon G \text{ is open in } X\},$$

and for $0 < a < \omega_1$ let

$$\mathfrak{G}_a = \Big\{G\colon G = \mathop{\mathsf{L}}_{n=1}^{\infty} G_n \text{ with } G_n \in \mathfrak{G}_{\beta_n},\ \beta_n < a\Big\},$$

where L stands for union if a is even and for intersection if a is odd. Then

$$\mathfrak{B} = \bigcup\{\mathfrak{G}_a\colon a < \omega_1\}.$$

The family $\mathfrak{B}$ may also be constructed as $\bigcup\{\mathfrak{F}_a\colon a < \omega_1\}$, where $\mathfrak{F}_a = \{X\setminus G\colon G \in \mathfrak{G}_a\}$.

For each $a < \omega_1$, whichever collection, $\mathfrak{G}_a$ or $\mathfrak{F}_a$, is closed under countable intersections, is called the *collection of Borel sets of multiplicative class* a; the other collection comprises the sets of *additive class* a. A set which is of multiplicative (additive) class a but not of additive (multiplicative) class a is said to be *of exact class* a. More details about Borel sets may be found in [3].

We represent the Cantor set by $\{0,1\}^N$, where $N = \{1,2,\ldots\}$; its metric is given by

$$d(x,y) = \sum_{i=1}^{\infty} |x(i)-y(i)|\cdot 2^{-i}.$$

We denote the point $(0,0,\ldots)$ by $\mathbf{0}$.

LEMMA 1. *Let (X,ϱ) be a non-empty, compact, 0-dimensional metric space. If F is closed in X and $0<\varepsilon<1$, then there is a function $f\colon X\to C$ such that*

$$f^{-1}(\mathbf{0}) = F \quad \text{and} \quad d\big(f(x),f(y)\big) \leqslant \varepsilon\varrho(x,y)$$

for every $x,y\in X$.

Proof. Assume that F is not open (or else the proof is easy). We can write

$$F = \bigcap_{i=0}^{\infty} G_i,$$

where the G_i's are clopen sets such that $G_0 = X$ and $G_i \supsetneq G_{i+1}$. Let

$$C_i = G_i\setminus G_{i+1} \quad \text{and} \quad \varepsilon_i = \varrho(C_i, G_{i+1}) > 0.$$

Let $k(0)$ be the least positive integer such that

$$\sum_{n=k(0)+1}^{\infty} 2^{-n} \leqslant \varepsilon\varepsilon_0;$$

for $i\geqslant 1$ let $k(i)$ be the least positive integer greater than $k(i-1)$ and such that

$$\sum_{n=k(i)+1}^{\infty} 2^{-n} \leqslant \varepsilon\varepsilon_i.$$

For $x\notin F$ pick the least i such that $x\in C_i$. Put $f(x)(n) = 0$ for $1\leqslant n \leqslant k(i)$, and $f(x)(n) = 1$ otherwise. For $x\in F$ let $f(x) = \mathbf{0}$. Then $f^{-1}(\mathbf{0}) = F$. To check the desired inequality, suppose (the other cases being trivial) that i is the least integer such that $x\in C_i$, j the least such that $y\in C_j$, and that $i>j$. Then

$$d\big(f(x),f(y)\big) = \sum_{n=1}^{\infty} |f(x)(n)-f(y)(n)|\cdot 2^{-n}$$
$$= \sum_{n=k(j)+1}^{\infty} |f(x)(n)-f(y)(n)|\cdot 2^{-n} \leqslant \sum_{n=k(j)+1}^{\infty} 2^{-n} \leqslant \varepsilon\varepsilon_j.$$

Since $y\in C_j$ and $x\in G_{j+1}$, we have $\varepsilon_j\leqslant\varrho(x,y)$, and so $d\big(f(x),f(y)\big) \leqslant \varepsilon\varrho(x,y)$.

In C, we now define sets M_α of multiplicative class α for each $\alpha < \omega_1$. For each α, a corresponding set $A_\alpha = C \setminus M_\alpha$ (of additive class α) is also defined. Let $M_0 = \{0\}$ and $A_0 = C \setminus M_0$.

Assume that, for each $\gamma < \alpha < \omega_1$, sets M_γ and $A_\gamma = C \setminus M_\gamma$ are defined. Partition N into infinitely many infinite sets $N_j = \{j_k \colon k = 1, 2, \ldots\}$, where $j_k < j_{k+1}$. Define $\sigma_j \colon N_j \to N$ by $\sigma_j(j_k) = k$. Then σ_j is an order isomorphism and $\sigma_j(j_k) \leqslant j_k$. Let $P_j = \{0, 1\}^{N_j}$ have the metric

$$d_j(x, y) = \sum_{i \in N_j} |x(i) - y(i)| \cdot 2^{-\sigma_j(i)}.$$

Then $\varphi_j \colon (P_j, d_j) \to (C, d)$ given by $\varphi_j(x)(k) = x(j_k)$ is an isometry, since

$$\begin{aligned} d\big(\varphi_j(x), \varphi_j(y)\big) &= \sum_{i=1}^{\infty} |\varphi_j(x)(i) - \varphi_j(y)(i)| \cdot 2^{-i} \\ &= \sum_{i=1}^{\infty} |x(j_i) - y(j_i)| \cdot 2^{-i} = \sum_{j_i \in N_j} |x(j_i) - y(j_i)| \cdot 2^{-i} \\ &= \sum_{i \in N_j} |x(i) - y(i)| \cdot 2^{-\sigma_j(i)} = d_j(x, y). \end{aligned}$$

For $\alpha = \gamma + 1$, let

$$M_\alpha = \{x \in C \colon x | N_j \in \varphi_j^{-1}(A_\gamma) = A_\gamma^j \text{ for } j = 1, 2, \ldots\}.$$

If α is a limit ordinal, let $\beta \to n(\beta)$ be a bijection from $[0, \alpha)$ to N and let

$$M_\alpha = \{x \in C \colon x | N_{n(\beta)} \in \varphi_{n(\beta)}^{-1}(A_\beta) = A_\beta^{n(\beta)} \text{ for all } \beta < \alpha\}.$$

In either case, let $A_\alpha = C \setminus M_\alpha$. In the first case, M_α is homeomorphic to

$$A_\gamma \times A_\gamma \times \ldots \subseteq C^{\aleph_0}$$

and, in the second case, to

$$\prod_{\beta < \alpha} A_\beta \subseteq C^{\aleph_0}.$$

Thus M_α is homeomorphic to a product of sets of multiplicative class α, and therefore [3] is of multiplicative class α.

LEMMA 2. *Let (X, ϱ) be a non-empty, compact, 0-dimensional metric space. If $M \subseteq X$ is of multiplicative class α and $0 < \varepsilon < 1$, then there is a map $f_\alpha \colon X \to C$ such that*

$$f_\alpha^{-1}(M_\alpha) = M \quad \textit{and} \quad d\big(f_\alpha(x), f_\alpha(y)\big) \leqslant \varepsilon \varrho(x, y)$$

for all $x, y \in X$. If M is of additive class α, then the analogous conclusion, with A_α replacing M_α, holds.

Proof. For each ordinal less than ω_1, the conclusion for additive sets follows from the earlier part simply by taking complements.

The case $\alpha = 0$ is Lemma 1. Suppose that Lemma 2 holds for all $\gamma < \alpha < \omega_1$. Since σ_j'' (as defined earlier) is an isometry, it follows from the definition of A_γ that if C_j is a set of additive class $< \alpha$ in X and $0 < \varepsilon_j < 1$, then there is a function $f_{\gamma(j)}\colon X \to P_j$ such that

$$f_{\gamma(j)}^{-1}(A_\gamma^j) = C_j \quad \text{and} \quad d_j\big(f_{\gamma(j)}(x), f_{\gamma(j)}(y)\big) \leqslant \varepsilon_j \varrho(x, y).$$

We now apply this fact.

If $\alpha = \gamma+1$, write

$$M = \bigcap_{j=1}^{\infty} C_j,$$

where C_j is of additive class γ. Write

$$\varepsilon = \sum_{j=1}^{\infty} \varepsilon_j, \quad \text{where } 0 < \varepsilon_j < 1.$$

For each j we have, by induction, the $f_{\gamma(j)}$ described above. Define $f_\alpha\colon X \to C$ by $f_\alpha(x)(i) = f_{\gamma(j)}(x)(i)$ for $i \in N_j$. Then $f_\alpha^{-1}(M_\alpha) = M$ and

$$\begin{aligned} d\big(f_\alpha(x), f_\alpha(y)\big) &= \sum_{n=1}^{\infty} |f_\alpha(x)(n) - f_\alpha(y)(n)| \cdot 2^{-n} \\ &= \sum_{j=1}^{\infty} \sum_{i \in N_j} |f_{\gamma(j)}(x)(i) - f_{\gamma(j)}(y)(i)| \cdot 2^{-i} \\ &\leqslant \sum_{j=1}^{\infty} \sum_{i \in N_j} |f_{\gamma(j)}(x)(i) - f_{\gamma(j)}(y)(i)| \cdot 2^{-\sigma_j(i)}, \end{aligned}$$

since $\sigma_j(i) \leqslant i$. Consequently,

$$d\big(f_\alpha(x), f_\alpha(y)\big) \leqslant \sum_{j=1}^{\infty} d_j\big(f_{\gamma(j)}(x), f_{\gamma(j)}(y)\big) \leqslant \sum_{j=1}^{\infty} \varepsilon_j \varrho(x, y) = \varepsilon \varrho(x, y).$$

If α is a limit ordinal, write

$$M = \bigcap \{C_{n(\beta)}\colon \beta < \alpha\},$$

where, as before, $\beta \to n(\beta)$ is a bijection between $[0, \alpha)$ and N, and where each $C_{n(\beta)}$ is of additive class $< \alpha$. Write

$$\varepsilon = \sum \{\varepsilon_{n(\beta)}\colon \beta < \alpha\}, \quad \text{where } 0 < \varepsilon_{n(\beta)} < 1.$$

For each $\beta < \alpha$ the induction hypothesis gives a function $f_\beta^{n(\beta)}\colon X \to P_{n(\beta)}$ such that

$$(f_\beta^{n(\beta)})^{-1}(A_\beta^{n(\beta)}) = C_{n(\beta)} \quad \text{and} \quad d_{n(\beta)}\big(f_{n(\beta)}(x), f_{n(\beta)}(y)\big) \leqslant \varepsilon_{n(\beta)} \varrho(x, y).$$

Define $f_a\colon X \to C$ by $f_a(x)(i) = f_\beta^{n(\beta)}(x)(i)$ for $i \in N_{n(\beta)}$. Then $f_a^{-1}(M_a) = M$, and the verification of the desired inequality proceeds as before.

THEOREM. *The Cantor set C contains sets of exact class a for each $a < \omega_1$.*

Proof. The set M_a is of multiplicative class a. If M_a were of additive class a, then Lemma 2 would give a function $f_a\colon (C, d) \to (C, d)$ such that

$$f_a^{-1}(M_a) = A_a \quad \text{and} \quad d\big(f_a(x), f_a(y)\big) \leqslant \tfrac{1}{2} d(x, y).$$

Since C is compact, the contraction mapping theorem provides a fixed point p for f; but then $p \in M_a \cap A_a = \emptyset$.

COROLLARY. *If X is a metrizable space containing C topologically, then X has Borel sets of exact class a for every $a < \omega_1$.*

Remarks. The corollary covers, for example, all metrizable spaces which contain a non-empty, dense in itself, completely metrizable subspace. These include, in particular, all uncountable complete, separable metric spaces. More generally, the corollary also covers, by a theorem of Elkin [1], every metric absolute $\aleph_0$-analytic set (and hence every metric absolute Borel set) which is not σ-discrete, i.e., which is not a countable union of discrete sets.

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